The $H_2$-norm is given by

$$
\|G\|_2 = \left( \frac{1}{\pi} \int_{-\infty}^{\infty} \text{tr} (G(i\omega) G^*(i\omega)) \, d\omega \right)^{1/2}
$$

**Interpretation:**

$H_2$ = "average" gain of system.

($\|G\|_\infty$ = peak gain)

**Trace:** trace = sum of all diagonal elements.

$$
\text{tr} (A) = \sum_{i=1}^{n} A_{ii}
$$

Assume we have an extended system

\[ E = \begin{bmatrix} W & G_u & \omega & \psi \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \]

With state space form

\[ 
\begin{align*}
    x &= Ax + Bu + Nw \\
    z &= Mx + Du \\
    y &= Cx + W
\end{align*}
\]

Such that $D^T E M D = [0 \ I]$

Then the controller which minimize

$\| G_{ec} \|_2$ is given by
\[
\begin{aligned}
\dot{x} &= A\hat{x} + Bu + NEy - C\hat{x} \\
\dot{u} &= -L\hat{x}
\end{aligned}
\]

with \( L = B^TS \) where \( S = S^T > 0 \) is the solution to

\[
A^TS + SA + M^TM - SBB^TS = 0
\]

Note: \( LQG, H_\infty, H_2 \) are all observer-based controllers on the form

\[
\begin{aligned}
\dot{x} &= A\hat{x} + Bu + KEy - C\hat{x} \\
\dot{u} &= -L\hat{x}
\end{aligned}
\]

where \( L \) is determined by solving a riccati equation.

For \( H_2 \& H_\infty \), we assume a state space model on innovation-form

\( \Rightarrow N \) is the optimal kalman gain \( K \) don't need to optimize the observer. (see book chapter 5).
10.3 Consider the extended system

\[ U(s) \xrightarrow{\varepsilon} Z_1(s) \xrightarrow{\frac{1}{s+1}} Z_2(s) \xrightarrow{\varepsilon} Y(s) \]

Show that
\[
\begin{align*}
\dot{x} &= -x + u + w \\
z &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\
y &= w
\end{align*}
\]
is a state space description of \( G_e \).

What is \( G_e \)?

3 outputs \( \rightarrow \) \( G_e \) is a 3x2 matrix
2 inputs \( \rightarrow \)

\( U \rightarrow Z_1, Z_2 \Rightarrow \)

\( \begin{align*}
G_{e11} &= 1 \\
G_{e21} &= \frac{1}{s+1} \\
G_{e31} &= 0 \\
G_{e12} &= 0 \\
G_{e22} &= \frac{1}{s+1} \\
G_{e32} &= 1
\end{align*} \)

\( \Rightarrow G_e = \begin{pmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{pmatrix} \)
Statespace to transfer function:

Laplace transform $\Rightarrow$

$$\begin{cases} \dot{X} = -X + U + W \\ Z_1 = U \\ Z_2 = X \Rightarrow \text{eliminate} \Rightarrow \begin{cases} Z_1 = U \\ Z_2 = \frac{1}{s+1} (U + W) \Rightarrow Y = W \\ G_{ec} = \begin{pmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{pmatrix} \end{cases} \end{cases}$$

b) An observer based controller is

$$\begin{cases} \dot{\hat{X}} = -\hat{X} + u + y \\ u = L\hat{X} \end{cases}$$

Determine $L$ such that

$\|G_{ec}(s)\|_{\infty}$ is minimized.

Compute $F_y(s)$.

$H_2$-controller:

The model can be written

$$\begin{align*}
\dot{X} &= Ax + Bu + Nw \\
Z &= Mx + Du \\
y &= Cx + w
\end{align*}$$

where $A = -1$ $B = 1$ $N = 1$ $M = \begin{pmatrix} 0 \end{pmatrix}$ $D = \begin{pmatrix} 1 \end{pmatrix}$ $C = 0$

and $D^T M D = [0 \ 1]$
Optimal H$_2$-controller is:

\[
\begin{align*}
\dot{x} &= Ax + Bu + N[y - Cx] \\
u &= -L\dot{x}
\end{align*}
\]

\Rightarrow Correct form for an H$_2$-controller.

**Determine $L$:**

\[L = B^T S\quad \text{where } S = S^T > 0 \text{ is the solution to}
\]

\[A^T S + SA + M^T M - SBB^T S = 0\]

\[-S - S + 1 - S^2 = 0 \Rightarrow S^2 + 2S - 1 = 0 \Rightarrow S = -1 \pm \sqrt{2} \quad (S > 0)\]

The optimal state feedback is

\[L = 1 \cdot (1 + \sqrt{2}) = -1 + \sqrt{2}\]

**Determine $F(x,y)$:**

In statespace the controller is

\[
\begin{align*}
\dot{x} &= -x + u + y \\
u &= (1 + \sqrt{2})\dot{x}
\end{align*}
\]
Laplace transform →

\[
\begin{align*}
\dot{X}(s) &= -X(s) + U(s) + Y(s) \\
U(s) &= \left(-1 + \frac{1}{2}\right) \dot{X}(s)
\end{align*}
\]

Eliminate \(\dot{X}(s)\) →

\[
\begin{align*}
\frac{s - U(s)}{1 + \frac{1}{2}} &= \frac{U(s)}{1 + \frac{1}{2}} + U(s) + Y(s) \Rightarrow \\
U(s) &= \frac{1 - \frac{1}{2}}{s + \frac{1}{2}} Y(s) \\
&= \frac{\sqrt{2}}{\sqrt{s + \frac{1}{2}}} F_y(s)
\end{align*}
\]

\(y\) What is \(\|G_{ec}(s)\|_2\) for this controller.

\[
\begin{align*}
\begin{pmatrix} \dot{z}_1(s) \\ \dot{z}_2(s) \end{pmatrix} &= \begin{pmatrix} 1 - \frac{\sqrt{2}}{s + \frac{1}{2}} \\ \frac{1}{s + 1} \left(1 + \frac{1 - \sqrt{2}}{s + \frac{1}{2}}\right) \end{pmatrix} W(s)
\end{align*}
\]
$$G_{ec}(s) = \frac{\left( 1 - \frac{1}{s + \sqrt{2}} \right)}{s + \sqrt{2}}$$

$$\|G_{ec}\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left( G_{ec}(i\omega) G_{ec}^*(i\omega) \right) d\omega$$

$$\text{tr} \left( G_{ec} G_{ec}^* \right) = \text{tr} \left( G_{ec}^* G_{ec} \right) = (\overline{G_{ec1}} \quad \overline{G_{ec2}})$$

$$= |G_{ec1}|^2 + |G_{ec2}|^2 \Rightarrow$$

$$\|G_{ec}\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{3 - 2\sqrt{2}}{\omega^2 + \frac{1}{\omega^2}} + \frac{1}{\omega^2 + \frac{1}{\omega^2}} \right) d\omega =$$

$$= \frac{2 - \sqrt{2}}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 + \frac{1}{\omega^2}} = \frac{2 - \sqrt{2}}{\pi} \left[ \frac{1}{\frac{1}{2} \ln \frac{\omega}{\sqrt{2}}} \right]_{-\infty}^{\infty} =$$

$$= \frac{2 - \sqrt{2}}{\pi} \cdot \frac{1}{\frac{1}{2} \pi} = \sqrt{2} - 1$$

$$\Rightarrow \|G_{ec}\| = \sqrt{\sqrt{2} - 1}$$
Let \( U(s) = -KY(s) \) (proportional controller).

Determine the \( K \) that minimize \( \| \text{Gecc} \|_2 \).

\[
\text{Gecc: } \quad F_y = -K \quad \Rightarrow
\]

\[
Z(s) = \begin{pmatrix} \frac{-K}{s+1} \\ \frac{1}{s+1}(1-K) \end{pmatrix} W
\]

we get

\[
\| \text{Gecc} \|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( -K \right)^2 + \frac{\| -K \|^2}{1+\omega^2} d\omega = \pi
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} K^2 d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\| -K \|^2}{1+\omega^2} d\omega
\]

\[
\rightarrow \infty \text{ if } K \neq 0
\]

\[
\Rightarrow \text{finite only when } K = 0
\]

\[
K = 0 \quad \Rightarrow
\]

\[
\| \text{Gecc} \|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega = \frac{1}{2\pi} \left[ \arctan \omega \right]_{-\infty}^{\infty} = \frac{1}{2}
\]

\[
\Rightarrow \| \text{Gecc} \|_2 = \frac{1}{\sqrt{2}} \approx \sqrt{12 - 1} \approx 0.71 \quad \approx 0.64
\]
An $H_\infty$ controller $F_y$ has been derived for
\[ G_c(s) = \frac{1+3}{1+0.15s + 2s^2} \]
such that $\| G_{rec} \|_\infty < \gamma = 2.5$
and $G_{rec} = \begin{pmatrix} W_u G_{ww} \\ -W_T T \\ W_s S \end{pmatrix}$
with $W_u = \text{constant}$
\[ W_T = \frac{s+3}{1+0.11s} \]
\[ W_s = \frac{s+3}{s} \]

\begin{align*}
\text{What is the order of } F_y \text{?}
\end{align*}

In state-space, the controller is
\[
\begin{align*}
\dot{x} &= Ax + Bu + NV[y-Cx] \\
\text{or } x &= Lx
\end{align*}
\]

Where $A, B, C, N$ comes from the state-space description of the extended system $H_y$, same order as the extended system.

If no simplifications can be done in the block diagram (pole-zero cancellations) then the minimum order is

\[ \text{Order } F_y = \text{order } G_y + \text{order } W_T + \text{order } W_u + \text{order } W_s = 2 + 1 + 0 + 1 = 4 \]
In this case, the weights have poles at 
$s=-10$, $s=0$ and zeros at $s=-3$

$G$ has poles at $s=0.075 \pm 0.703i$ and a zero at $s=1$

$\Rightarrow$ no cancellations possible $\Rightarrow F_y$ has order 4.

---

by Disturbances at frequencies below $\omega_0$ rad/s
should be damped by at least a factor of $10_0$.

Is this fulfilled for the nominal system?

(i.e., assuming $G_y$ is a correct model)

We know that

$\|W_s\|_\infty \leq \|G_y\|_\infty < \gamma \Rightarrow |G_y(i\omega)| < \gamma$

$\Rightarrow |G_y(i\omega)| < \frac{\gamma}{|W_s(i\omega)|} = \frac{2.5}{|1\omega+3|} = \frac{2.5\omega}{\sqrt{\omega^2+9}}$

$|G_y(0.1)| = \frac{0.25}{\sqrt{9.01}} < \frac{0.25}{3} < 0.1$

$\Rightarrow$ Yes, the damping is OK for the nominal system.
To reduce the order of the controller
some fast dynamics were neglected.
If the true model is given by
\[ G_{\text{true}} = \frac{1}{1 + 0.1s} \]

is the system stable with the same Fig?

**Robustness criterion:**

Let \[ G_0 = (1 + \Delta G) G \]

if \[ |\Delta G(i\omega)| < \frac{1}{|T(i\omega)|} \]

then the system is stable anyway.

**Determine \( \Delta G \):**

\[ (1 + \Delta G) G = \frac{1}{1 + 0.1s} \quad \Rightarrow \]

\[ 1 + \Delta G = \frac{1}{1 + 0.1s} \quad \Rightarrow \quad \Delta G = \frac{\frac{1}{1 + 0.1s} - 1}{1 + 0.1s} = \frac{-0.1s}{1 + 0.1s} \]

**\( |T(i\omega)| \):**

We know that \[ \|W_T\|_{\infty} < \|G(e)\|_{\infty} < \gamma = 2.5 \]

\[ \Rightarrow \quad \left| \frac{1}{T(i\omega)} \right| > \left| \frac{W_T(i\omega)}{\gamma} \right| \quad \Rightarrow \quad \text{Stable if} \]

\[ |\Delta G(i\omega)| < \left| \frac{W_T(i\omega)}{\gamma} \right| < \left| \frac{1}{T(i\omega)} \right| \]
\( \Rightarrow \) stable if \( \Re \{ \Delta G_c(j\omega) \} / |W_T(j\omega)| < 1 \)

We have

\[
\Re \{ \Delta G_c(j\omega) \} = \frac{0.25 \omega}{\sqrt{1 + 0.01\omega^2}}
\]

and

\[
|W_T(j\omega)| = \frac{\sqrt{\omega^2 + 3}}{\sqrt{1 + 0.01\omega^2}}
\]

Same denominator \( \Rightarrow \) stable if

\[
0.25 \omega < \sqrt{\omega^2 + 3}
\]

which is trivially true. (since \( \sqrt{\omega^2 + 3} > \omega > 0.25\omega \))

Hence the system is stable even with the model error.

Note: Since \( H^0 \) allows us to shape \( K \), we can easily make systems robust!
7.3 Disturbances & noise should be dampened by a factor of 10 for freqs below 0.1 rad/s resp above 2 rad/s.

Constant disturbances should be dampened by at least a factor 100.

a) Formulate this as requirements on $S$ & $T$.

\[ \overline{\sigma}(S(iw)) \text{ is the maximum amplification at freq } w. \Rightarrow \]

\[ \overline{\sigma}(S(iw)) \leq 0.1 \text{ for } w < 0.1 \Rightarrow \text{ok damping} \]

For the static case we need

\[ \overline{\sigma}(S(0)) \leq 0.01 \]
For noise we need
\[ \bar{\sigma}(T(i\omega)) < 0.1 \quad \omega > 2 \text{ rad/s} \]

\[ \bar{\sigma}(T) \]

\[ 1 \]

\[ 0,1 \]

\[ T \]

\[ \rightarrow \]

\[ \omega \]

b, translate the requirements on \( S \) & \( T \) to requirements on the loopgain.

\[ S = (I + G_F)^{-1} \quad T = G_F(I + G_F)^{-1} \]

Some singular value inequalities etc.

1) \[ \bar{\sigma}(A^{-1}) = 1/\bar{\sigma}(A) \]

2) \[ \bar{\sigma}(A) - I \leq \bar{\sigma}(I + A) \leq \bar{\sigma}(A) + 1 \]

3) \[ \bar{\sigma}(AB) \leq \bar{\sigma}(A) \bar{\sigma}(B) \]

4) \[ \bar{\sigma}(A) - \bar{\sigma}(B) \leq \bar{\sigma}(A + B) \leq \bar{\sigma}(A) + \bar{\sigma}(B) \]
Apply \( \textcircled{1} \) to \( S \) →

\[
\bar{\sigma}(S) < 0.1 \quad \text{and} \quad \omega < 0.1 \Rightarrow
\]

\[
\frac{1}{\bar{\sigma}(I + GF)} < 0.1 \Rightarrow \bar{\sigma}(I + GF) > 10 \quad \text{and} \quad \omega < 0.1
\]

Now from \( \textcircled{1} \) we get

\[
\bar{\sigma}(GF) > 11 \Rightarrow \bar{\sigma}(I + GF) > 10 \quad \text{and} \quad \omega < 0.1
\]

For \( \text{I} \): First we apply \( \textcircled{3} \) to get

\[
\bar{\sigma}(T) \leq \bar{\sigma}(GF) \cdot \bar{\sigma}(I + GF)^{-1} = \frac{\bar{\sigma}(GF)}{\bar{\sigma}(I + GF)}
\]

Worst case when \( \bar{\sigma}(I + GF) \) small,

we use \( \textcircled{4} \) to get

\[
\bar{\sigma}(I + GF) \geq 1 - \bar{\sigma}(GF) \Rightarrow
\]

\[
\bar{\sigma}(T) < \frac{\bar{\sigma}(GF)}{1 - \bar{\sigma}(GF)} \quad \text{so we need}
\]

\[
\frac{\bar{\sigma}(GF)}{1 - \bar{\sigma}(GF)} < 0.1 = \frac{0.1}{1.1} \Rightarrow \bar{\sigma}(GF) < \frac{0.1}{1.1} \Rightarrow \bar{G}(GF) < 0.091
\]
Hence we get:

\[ \sigma(G,F) \]

And also \( \leq (\zeta(0)\varphi(0)) \geq 101 \)
Formulate the requirements using $Wt$ and $Ws$

We get $\|W_t\|_\infty < 1$ with $|W_t(i\omega)| > \frac{1}{\alpha_1}$ \( w > 2 \)

$\|W_s\|_\infty < 1$ with $|W_s(i\omega)| > \frac{1}{\alpha_1}$ \( w < 1 \)