Theory: H_2-control:

The H_2-norm is given by

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}\left(G(i\omega) \cdot G^*(i\omega) \right) d\omega \right)^{1/2}$$

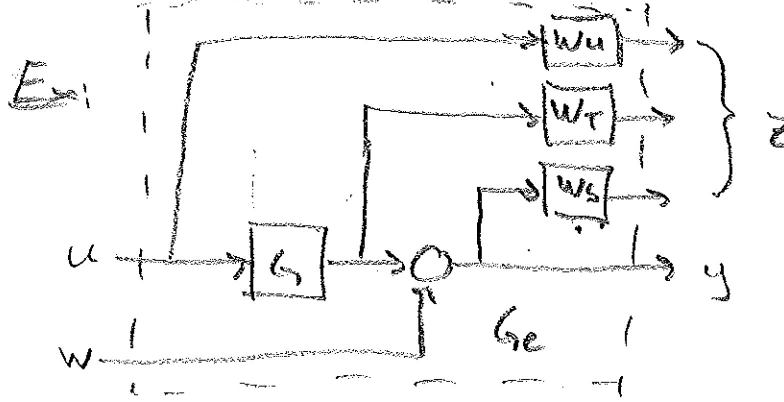
Interpretation:
H_2 = "Average" gain of system.
(H_∞ = peak gain)

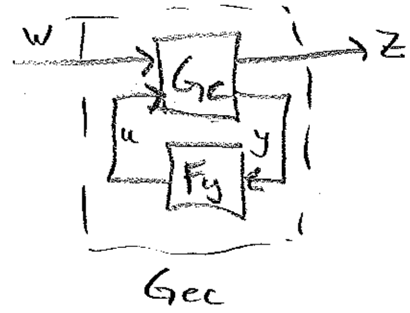
tr = trace = sum of all diagonal elements.

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

Assume we have an extended system

Ex:





With state space form

$$\dot{x} = Ax + Bu + Nw$$
$$z = Mx + Du$$
$$y = Cx + w$$

Such that $D^T[M \; D] = [0 \; I]$

Then the controller which minimize

$\|G_{ec}\|_2$ is given by

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + N[y - C\hat{x}] \\ u = -L\hat{x} \end{cases}$$

with $L = B^T S$ where $S = S^T \geq 0$ is the solution to

$$A^T S + SA + M^T M - SBB^T S = 0$$

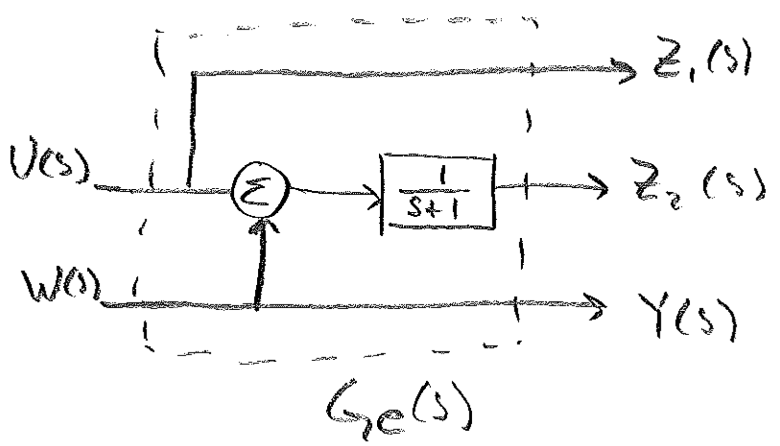
<u>Note:</u> LQG, H_∞, H_2 are all observer-based controllers on the form

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K[y - C\hat{x}] \\ u = -L\hat{x} \end{cases}$$

where L is determined by solving a riccatti equation.

For H_2 & H_∞, we assume a state space model on innovation-form
$\Rightarrow$ N is the optimal kalman gain K $\Rightarrow$ don't need to optimize the observer. (see book chapter 5).

10.3 Consider the extended system



a) Show that

$$\dot{x} = -x + u + w$$

$$z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$y = w$$

is a state space description of G_e.

<u>What is G_e!</u>

3 outputs, 2 inputs $\Big\} \Rightarrow G_e = 3 \times 2$ - matrix

$U \rightarrow \begin{matrix} Z_1 \\ Z_2 \\ Y \end{matrix} \Rightarrow$
$G_{e11} = 1$
$G_{e21} = \frac{1}{s+1}$
$G_{e31} = 0$

$$\Rightarrow G_e = \begin{pmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{pmatrix}$$

$W \rightarrow \begin{matrix} Z_1 \\ Z_2 \\ Y \end{matrix} \Rightarrow$
$G_{e12} = 0$
$G_{e22} = \frac{1}{s+1}$
$G_{e32} = 1$

Statespace to transferfunction:

Laplace transform →

$$\begin{cases} sX = -X + U + W \\ Z_1 = U \\ Z_2 = X \\ Y = W \end{cases} \Rightarrow \underset{X}{\text{eliminate}} \Rightarrow \begin{cases} Z_1 = U \\ Z_2 = \frac{1}{s+1}(U+W) \\ Y = W \end{cases} \Rightarrow$$

$$G_e = \begin{pmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{pmatrix}$$

b) An observer based controller is

$$\begin{cases} \dot{\hat{x}} = -\hat{x} + u + y \\ u = -L\hat{x} \end{cases}$$

Determine L such that

$\|G_{ec}\|_2^2$ is minimized.

Compute $F_y(s)$.

H_2-controller:

The model can be written

$$\begin{aligned} \dot{x} &= Ax + Bu + Nw \\ z &= Mx + Du \\ y &= Cx + w \end{aligned} \qquad A = -1 \quad B = 1 \quad N = 1 \quad M = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = 0$$

and $\underbrace{D^T[M\ D] = [0\ 1]}_{\checkmark}$

Optimal H_2-controller is

$$\begin{cases} \dot{\hat{x}} = Ax + Bu + N[y - C\hat{x}] \\ u = -L\hat{x} \end{cases} = \begin{cases} \dot{\hat{x}} = -x + u + y \\ u = -L\hat{x} \end{cases}$$

$\Rightarrow$ Correct form for an H_2-controller.

<u>Determine L:</u>

$L = B^T S$ where $S = S^T \geq 0$ is the solution to

$$A^T S + SA + M^T M - SBB^T S = 0$$

$$-S - S + 1 - S^2 = 0 \Rightarrow$$

$$S^2 + 2S - 1 = 0 \Rightarrow$$

$$S = -1 \pm \sqrt{2}$$

$$(\nwarrow S \geq 0)$$

The optimal <u>state feedback</u> is

$$L = 1 \cdot (-1 + \sqrt{2}) = -1 + \sqrt{2}$$

<u>Determine $F_y(s)$:</u>

In statespace the controller is

$$\dot{\hat{x}} = -\hat{x} + u + y$$
$$u = (-1 + \sqrt{2})\hat{x}$$

Laplace transform $\rightarrow$

$$\begin{cases} s\hat{X}(s) = -\hat{X}(s) + U(s) + Y(s) \\ U(s) = -(-1+\sqrt{2})\,\hat{X}(s) \end{cases}$$

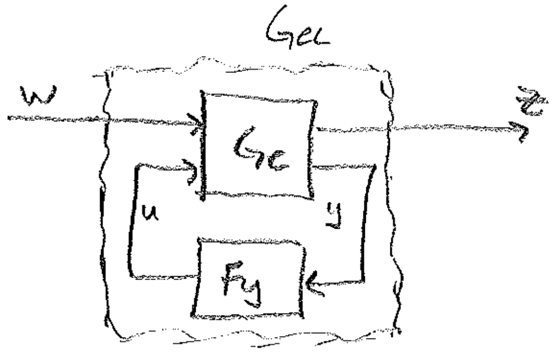
Eliminate $\hat{X}(s)$ $\longrightarrow$

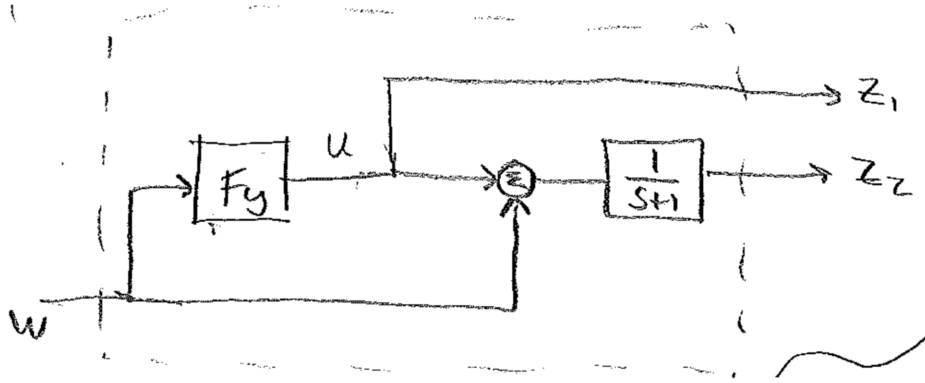
$$s\,\frac{-U(s)}{-1+\sqrt{2}} = \frac{U(s)}{-1+\sqrt{2}} + U(s) + Y(s) \Rightarrow$$

$$U(s) = \underbrace{\frac{1-\sqrt{2}}{s+\sqrt{2}}}_{F_y(s)}\,Y(s)$$

c) What is $\|G_{ec}\|_2$ for this controller.

G_{ec}:





$$\Rightarrow \begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix} = \overbrace{\begin{pmatrix} \frac{1-\sqrt{2}}{s+\sqrt{2}} \\ \frac{1}{s+1}\left(1 + \frac{1-\sqrt{2}}{s+\sqrt{2}}\right) \end{pmatrix}}^{G_{ec}(s)} W(s)$$

$$G_{ec}(s) = \begin{pmatrix} \frac{1-\sqrt{2}}{s+\sqrt{2}} \\ \frac{1}{s+\sqrt{2}} \end{pmatrix}$$

$$\|G_{ec}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left(G_{ec}(i\omega)\, G_{ec}(i\omega)^*\right) d\omega \; :$$

$$tr\left(G_{ec} G_{ec}^*\right) = tr\left(G_{ec}^* G_{ec}\right) = \begin{pmatrix}\overline{G_{ec1}} & \overline{G_{ec2}}\end{pmatrix}\begin{pmatrix} G_{ec1} \\ G_{ec2}\end{pmatrix}$$

$$= |G_{ec1}|^2 + |G_{ec2}|^2 \Rightarrow$$

$$\|G_{ec}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{3-2\sqrt{2}}{\omega^2+2} + \frac{1}{\omega^2+2}\right) d\omega =$$

$$= \frac{2-\sqrt{2}}{\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2+2} = \frac{2-\sqrt{2}}{\pi}\left[\frac{1}{\sqrt{2}}\arctan\frac{\omega}{\sqrt{2}}\right]_{-\infty}^{\infty} =$$

$$= \frac{2-\sqrt{2}}{\pi}\cdot\frac{1}{\sqrt{2}}\pi = \sqrt{2}-1$$

$$\Rightarrow \|G_{ec}\|_2 = \sqrt{\sqrt{2}-1}$$

d) Let $U(s) = -K\,Y(s)$ (proportional controller).

Determine the K that minimize $\|G_{ec}\|_2$

<u>G_{ec}:</u> $F_y = -K \Rightarrow$

$$Z(s) = \underbrace{\begin{pmatrix} -K \\ \frac{1}{s+1}(1-K) \end{pmatrix}}_{G_{ec}} w$$

we get

$$\|G_{ec}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(|-K|^2 + \frac{|1-K|^2}{1+\omega^2}\right)d\omega =$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} K^2 d\omega}_{=\infty \text{ if } K\neq 0} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|1-K|^2}{1+\omega^2}d\omega$$

$\Rightarrow$ finite only when $K = 0$

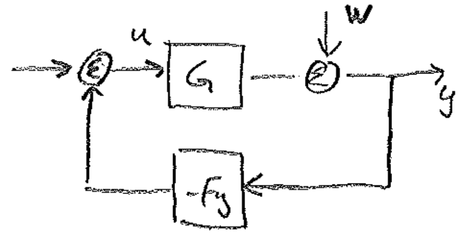
<u>$K=0$</u> $\Rightarrow$

$$\|G_{ec}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\omega^2}d\omega = \frac{1}{2\pi}\left[\arctan\omega\right]_{-\infty}^{\infty} = \frac{1}{2}$$

$$\Rightarrow \|G_{ec}\|_2 = \frac{1}{\sqrt{2}} > \sqrt{\sqrt{2}-1}$$

$\approx 0{,}71 \qquad \approx 0{,}64$

10.9 An H_∞-controller F_y has been derived for



$$G(s) = \frac{1+s}{1+0{,}3s+2s^2}$$

such that $\|G_{ec}\|_\infty < \gamma = 2{,}5$

and $G_{ec} = \begin{pmatrix} W_u G_{wu} \\ -W_T T \\ W_S S \end{pmatrix}$

with $W_u =$ constant

$$W_T = \frac{s+3}{1+0{,}1s}$$

$$W_S = \frac{s+3}{s}$$

a) What is the order of F_y?

In statespace, the controller is

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + N[y - C\hat{x}] \\ u = -L\hat{x} \end{cases}$$

Where A, B, C, N comes from the statespace description of the <u>extended</u> system $\Rightarrow$ F_y same order as the extended system.

If no simplifications can be done in the block diagram (pole-zero cancellations) then the minimum order is

Order F_y = order G + order W_T + order W_u + order W_S $= 2+1+0+1 = 4$

In this case, the weights have poles at

$s = -10$, $s = \infty$ and zeros at $s = -3$

G has poles at $-0,075 \pm 0,703\,i$ and a zero at $s = 1$

$\Rightarrow$ no cancellations possible $\Rightarrow$ F_y has order 4.

---

b) Disturbances at frequencies below 0.1 rad/s should be damped by at least a factor of 10.

Is this fulfilled for the nominal system?
(i.e. assuming G is a correct model)

We know that

$$\|W_S S\|_\infty \leq \|G_{ec}\|_\infty < \gamma = 2{,}5 \Rightarrow |S(i\omega)|\,|W_S(i\omega)| < \gamma$$

$$\Rightarrow \quad |S(i\omega)| < \frac{\gamma}{|W_S(i\omega)|} = \left|\frac{2.5\, i\omega}{i\omega + 3}\right| = \frac{2.5\,\omega}{\sqrt{\omega^2+9}}$$

$$|S(i0{,}1)| = \frac{0{,}25}{\sqrt{9{,}01}} < \frac{0{,}25}{3} < 0{,}1$$

$\Rightarrow$ Yes, the damping is OK for the nominal system.

c) To reduce the order of the controller some fast dynamics were neglected.
If the true model is given by

$$G_0 = G \cdot \frac{1}{1+0{,}1s}$$

is the system stable with the same F_y?

<u>Robustness criterion:</u>

let $G_0 = (1+\Delta G)\,G$

if $|\Delta G(i\omega)| < \frac{1}{|T(i\omega)|}$ $\forall \omega$ then

the system is stable anyway.

<u>Determine ΔG:</u>

$$(1+\Delta G)\,G = \frac{1}{1+0{,}1s}\,G \Rightarrow$$

$$1+\Delta G = \frac{1}{1+0{,}1s} \Rightarrow \Delta G = \frac{1}{1+0{,}1s} - 1 = \frac{-0{,}1s}{1+0{,}1s}$$

<u>$|T^{-1}(i\omega)|$:</u>

We know that $\|W_T T\|_\infty < \|G_{ec}\|_\infty < \gamma = 2{,}5$

$$\Rightarrow \left|\frac{1}{T(i\omega)}\right| > \left|\frac{W_T(i\omega)}{\gamma}\right| \Rightarrow \text{Stable if}$$

$$|\Delta G(i\omega)| < \left|\frac{W_T(i\omega)}{\gamma}\right| < \left|\frac{1}{T(i\omega)}\right|$$

$\Rightarrow$ stable if $\gamma|\Delta G(i\omega)| < |W_T(i\omega)|$

We have

$$\gamma|\Delta G(i\omega)| = \frac{0{,}25\,\omega}{\sqrt{1+0{,}01\omega^2}}$$

and $$|W_T(i\omega)| = \frac{\sqrt{\omega^2+3}}{\sqrt{1+0{,}01\omega^2}}$$

Same denominator $\Rightarrow$ stable if

$$0{,}25\,\omega < \sqrt{\omega^2+3}$$

which is trivially true. (since $\sqrt{\omega^2+3} > \omega > 0{,}25\omega$)

Hence the system is stable even with the model error.

<u>Note:</u> Since H_∞ allow us to shape T, we can easily make systems robust!

7.3 Disturbances & noise should be dampened by a factor of 10 for freqs below

0,1 rad/s resp above 2 rad/s.

Constant disturbances should be dampened by at least a factor 100.

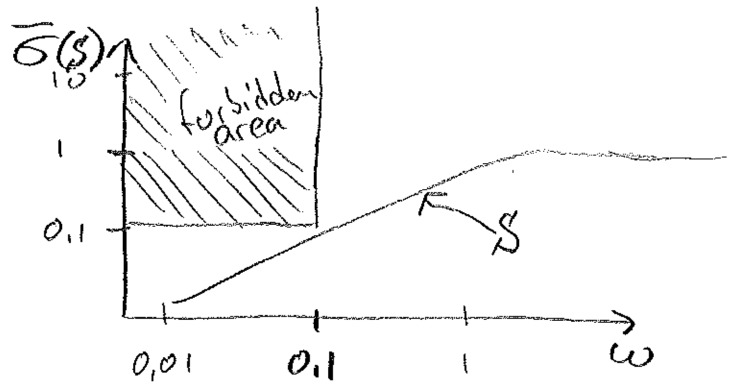
a) Formulate this as requirements on S & T.

$\bar{\sigma}(S(i\omega))$ is the maximum amplification at freq ω. $\Rightarrow$

$\bar{\sigma}(S(i\omega)) \leq 0{,}1$ for $\omega < 0{,}1$ $\Rightarrow$ ok damping

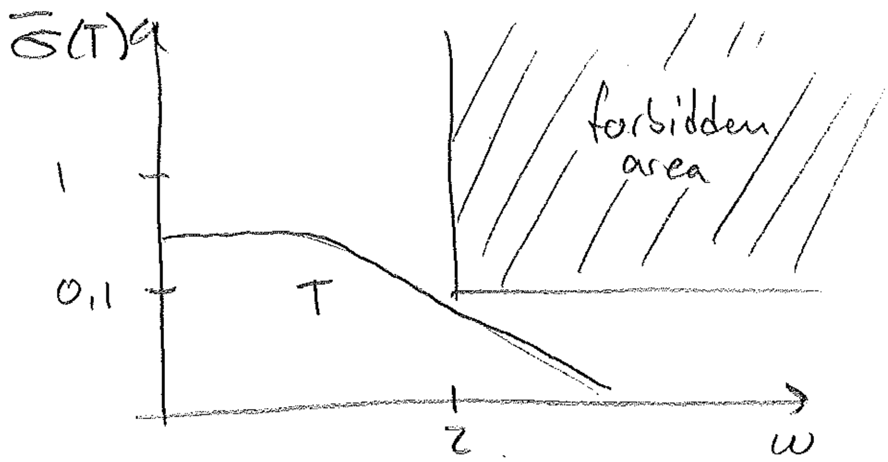
For the static case we need

$\bar{\sigma}(S(0)) \leq 0{,}01$



For noise we need

$$\bar{\sigma}(T(i\omega)) < 0.1 \qquad \omega > 2 \text{ rad/s}$$



b, translate the requirements on S & T to requirements on the loopgain.

$$S = (I + GF)^{-1} \qquad T = GF(I+GF)^{-1}$$

Some singular value inequalities etc.

(I) $\bar{\sigma}(A^{-1}) = 1/\underline{\sigma}(A)$

(II) $\underline{\sigma}(A) - 1 \leq \underline{\sigma}(I+A) \leq \underline{\sigma}(A) + 1$

(III) $\bar{\sigma}(AB) \leq \bar{\sigma}(A)\,\bar{\sigma}(B)$

(IV) $\underline{\sigma}(A) - \bar{\sigma}(B) \leq \underline{\sigma}(A+B) \leq \underline{\sigma}(A) + \bar{\sigma}(B)$

Apply (I) to S $\Rightarrow$

$\bar{\sigma}(S) < 0{,}1 \quad \omega < 0{,}1 \Rightarrow$

$\frac{1}{\underline{\sigma}(I+GF)} < 0{,}1 \Rightarrow \underline{\sigma}(I+GF) > 10 \quad \omega < 0{,}1$

Now from (II) we get

$\underline{\sigma}(GF) > 11 \Rightarrow \underline{\sigma}(I+GF) > 10 \quad \omega < 0{,}1$

For T: First we apply III to get

$$\bar{\sigma}(T) \leq \bar{\sigma}(GF) \cdot \bar{\sigma}((I+GF)^{-1}) = \frac{\bar{\sigma}(GF)}{\underline{\sigma}(I+GF)}$$

Worst case when

$\underline{\sigma}(I+GF)$ small.

We use (IV) to get

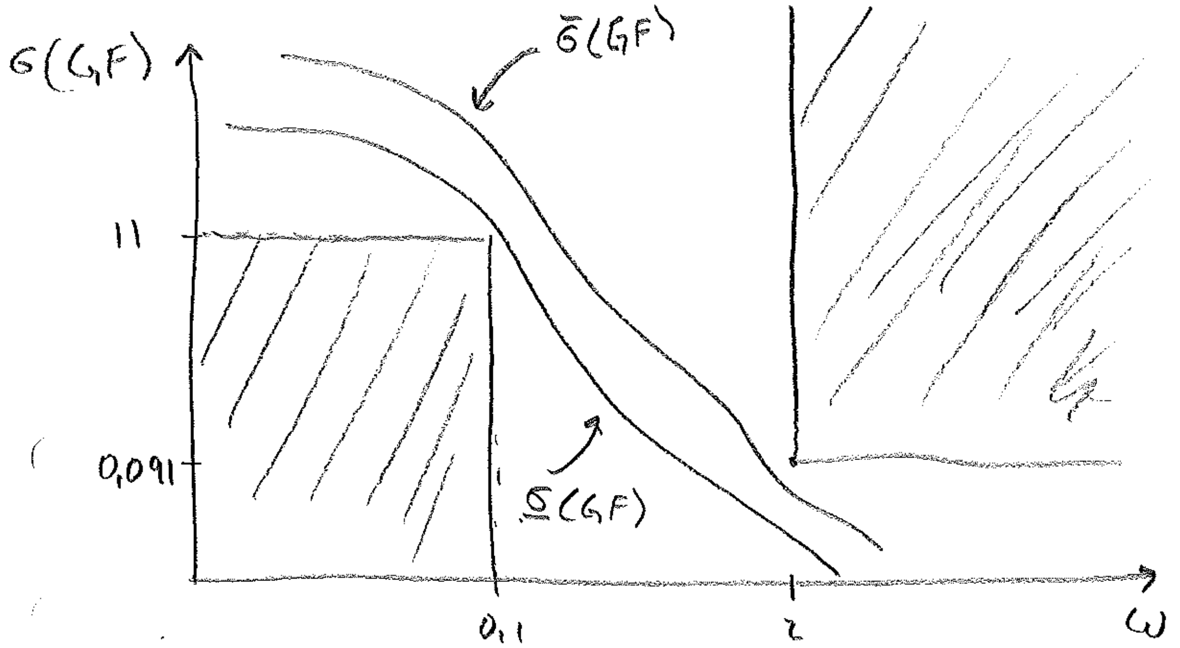
$\underline{\sigma}(I+GF) \geq 1 - \bar{\sigma}(GF) \Rightarrow$

$\bar{\sigma}(T) < \frac{\bar{\sigma}(GF)}{1-\bar{\sigma}(GF)}$ so we need

$\frac{\bar{\sigma}(GF)}{1-\bar{\sigma}(GF)} < 0{,}1 = 1{,}1 \cdot \bar{\sigma}(GF) < 0{,}1 \Rightarrow$

$\bar{\sigma}(GF) < \frac{0{,}1}{1{,}1} \approx 0{,}091$

Hence we get:



And also $\underline{\sigma}(G(0)F(0)) \geq 101$

c) Formulate the requirements using $\|\cdot\|_\infty$ and W_T & W_S

We get $\|W_T T\|_\infty \leq 1$ with $|W_T(i\omega)| > \frac{1}{0.1}$ $\omega > 2$

$\|W_S S\|_\infty < 1$ with $|W_S(i\omega)| > \frac{1}{0.1}$ $\omega < 0.1$