REGLERTEKNIK

School of Electrical Engineering, KTH

EL2520 Control Theory and Practice – Advanced Course

Exam (tentamen) 2012–08–14, kl 08.00–13.00

Aids: The course book for EL2520 (advanced course) and EL1000/EL1100 (basic course), copies of slides from this year's lectures, mathematical tables and pocket calculator. Note that exercise materials (övningsuppgifter, ex-tentor och lösningar) are NOT allowed.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be justified.
Lacking justification will results in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Only write on one side of each sheet.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have marked.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

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Resultat: Will be posted no later than September 4, 2012.

Good Luck!

1. a) Consider the system

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x$$

where α is a scalar constant.

i) How many inputs, outputs, and states does the system have? (1p)

ii) Compute the transfer matrix of the system, $G(s)$, for $\alpha = 1$. (1p)

iii) Compute the poles and zeros of the system for $\alpha = 1$. (2p)

b) We want to use decentralized control on the system

$$G(s) = \frac{1}{s+1} \begin{pmatrix} \frac{1}{s+1} & -1 \\ 2 & \frac{2}{s+1} \end{pmatrix}, \qquad (1)$$

with desired cross-over frequency $\omega_c = 10$ rad/s. Which inputs and outputs are suitable to pair? Motivate. (3p)

Hint:

$$G(i\omega_c) \approx \begin{pmatrix} -0.01 - 0.002i & -0.01 + 0.1i \\ 0.02 - 0.2i & -0.02 + 0.004i \end{pmatrix},$$
$$G(i\omega_c)^{-1} \approx \begin{pmatrix} 1.02 + 0.0001i & 0.49 + 5i \\ -0.98 - 10i & 0.5 \end{pmatrix}.$$

c) Consider the system

$$y = Gu + G_d d,$$

where $G(s) = \frac{1}{0.1s+1}$ and $G_d(s) = \frac{4}{s+6}$. The disturbance $d(t)$ is constant and lies in the interval $-4 < d < 4$. The control signal is constrained to satisfy $|u(t)| < 1$. Is it possible to use $u(t)$ to eliminate the influence of the disturbance in stationarity? (3p)

2. Consider a feedback controlled system with

$$G(s) = \frac{s-2}{s}$$

a) The system is required to reject disturbances for all $\omega \leq \omega_B$. Determine a reasonable value for ω_B by using the rule of thumb. (1p)

b) Determine a proportional controller $F(s) = K$ that satisfies the specification in a) and guarantees internal stability. (3p)

c) Calculate $\|S\|_\infty$ for the controller in b). If you are unable to solve b), you may use $K = -\frac{2}{3}$. (3p)

d) Let

$$W_s(s) = \frac{s + \omega_0 S_0}{S_0 s}$$

where S_0 is $\|S\|_\infty$ from c). Find the biggest ω_0 such that $\|W_S S\|_\infty \leq 1$ for the controller in b). If you are unable to solve b) and c), you may use $K = -\frac{1}{2}$ and $S_0 = 2$. (3p)

3. (a) Consider the interconnection in Figure 1 and show that it can be written as the interconnection of a linear system with the uncertainty

$$\Delta = \text{diag}(\Delta_G,\, \Delta_K)$$

Develop a stability criterion (expressed in terms of G, K and Δ) for the closed-loop system based on the small-gain theorem. (4p)

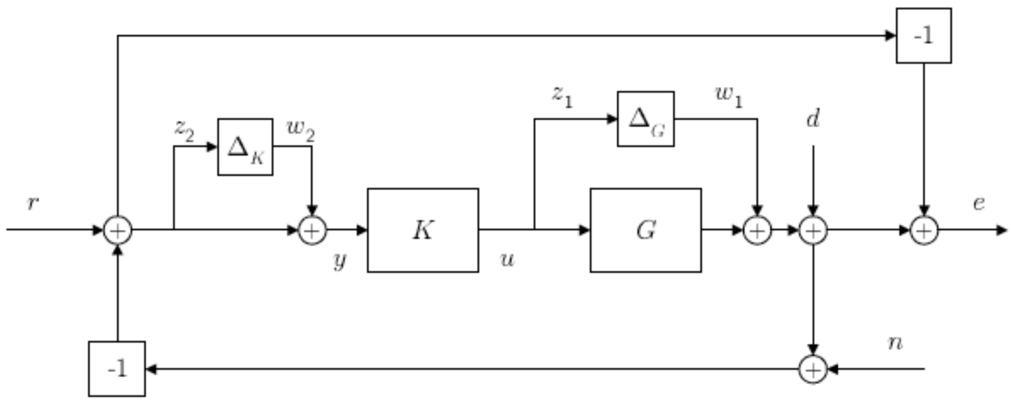

Figure 1: Closed-loop system with uncertainties.

(b) When we have parametric uncertainties, *i.e.* when the structure of the system is well-known, but the precise parameter values are not, the uncertainties will typically appear directly in the differential equations describing the system dynamics. As an example, we will consider the system

$$\dot{x} = \begin{pmatrix} -1+\delta_1 & 0 \\ 0 & -2+\delta_2 \end{pmatrix} x \tag{2}$$

Show that we can put this system into standard form, *i.e.* write it as the interconnection of a linear system on the form

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx \\ w &= \Delta z \end{aligned}$$

and the uncertainty block

$$\Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

Derive a stability criterion for the closed loop system based on the small gain theorem. What is the maximum value of δ_1 and δ_2 that your criterion can allow? (4p)

(c) The stability criteria derived in (b) results in conditions of the type

$$|\delta_i| < c_i, \quad i = 1, 2$$

for some positive scalars c_i. Derive the necessary and sufficient conditions on δ_1 and δ_2 for (2) to be stable and comment on the conservatism of the stability criteria derived in (b). (2p)

4. **(a)** Consider the discrete-time first-order system

$$x_{k+1} = ax_k + bu_k$$

where a and b are scalars.

Initially the system is controlled using a Linear Quadratic Regulator minimizing the cost

$$J(x_0, u) = \sum_{t=0}^{\infty} Q_1 x_t^2 + \sum_{t=0}^{\infty} Q_2 u_t^2$$

for given scalars Q_1 and Q_2.

Compute the solution to the Riccati equation, S, and the optimal state-feedback law $u_k = -Lx_k$. (2p)

Remark 1 *Note that the system is in discrete-time and so the LQR controller should be designed using the discrete-time Riccati equation, c.f. section 9.5 of the course book.*

(b) The LQR controller was then replaced by a Model Predictive Controller solving the following optimization problem

$$\begin{aligned} \min_u \quad & Q_N x_{k+N_p}^2 + \sum_{t=k}^{k+N_p-1} Q_1 x_t^2 + \sum_{t=k}^{k+N_p-1} Q_2 u_t^2 \\ \text{subject to } & x_{k+1} = ax_k + bu_k, \quad \forall k. \end{aligned}$$

i) Considering the prediction horizon $N_p = 1$, compute the control action u_0 as a function of the initial condition x_0 and the terminal weight Q_N. (3p)

ii) As seen in the previous exercise, solving the optimization problem for a given x_k results in a state-feedback law $u_k = -L_{MPC}x_k$. How could one choose the terminal weight Q_N so that the MPC recovers the performance of the LQR controller? (2p)

(c) Now consider the MPC controller solving the constrained optimization problem

$$\begin{aligned} \min_u \quad & Q_N x_{k+N_p}^2 + \sum_{t=k}^{k+N_p-1} Q_1 x_t^2 + \sum_{t=k}^{k+N_p-1} Q_2 u_t^2 \\ \text{subject to } & |u_k| < 1, \quad \forall k \\ & x_{k+1} = ax_k + bu_k, \quad \forall k. \end{aligned}$$

For the prediction and control horizon $N_p = 1$, translate the MPC problem into a Quadratic Programming (QP) problem

$$\begin{aligned} \min_u \quad & u^\top H u + h^\top u \\ \text{subject to } & Lu < b. \end{aligned}$$

That is, determine H, h, L and b. (3p)

5. In the course, we have worked with linear-quadratic control of linear systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ z(t) &= Cx(t) \\ y(t) &= x(t) \end{aligned} \tag{3}$$

and worked out the optimal controller that minimizes a criterion on the form

$$J = \int_{s=0}^{T} x(s)^T Q x(s) + u(s)^T R u(s)\, ds \tag{4}$$

i.e. a criterion where we penalize the (weighted) deviation of x and u from zero. The optimal solution is a state feedback

$$u(t) = -L_{\text{LQ}} x(t)$$

where the optimal gains can be found by solving an algebraic Riccati equation.

If we would like the output of (3) to track a *constant* reference signal r, then it is more natural to try to minimize a criterion on the form

$$J' = \int_{s=0}^{T} (x(t) - x^\star)^T Q (x(t) - x^\star) + (u(t) - u^\star)^T R (u(t) - u^\star)\, dt \tag{5}$$

where $x^\star$ is the stationary state vector for which $z^\star = Cx^\star = r$ and $u^\star$ is the constant control input that attains $x(t) = x^\star$ in stationarity. In this quiz, your task will be to work out the optimal controller and compare it with the "fix" that we used in the lectures. Specifically

(a) Show that the optimal control problem associated with (5) can be written on the same form as the standard LQ control problem (3-4), if we consider new variables $\tilde{x}(t) = x(t) - x^\star$, $\tilde{u}(t) = u(t) - u^\star$, $\tilde{y}(t) = y(t) - y^\star$ and $\tilde{z}(t) = z(t) - z^\star$. Show that the optimal controller is on the form

$$\tilde{u}(t) = -L\tilde{x}(t)$$

and describe how we can find the optimal state feedback gain L. (4p)

(b) Show that $x^\star$ and $u^\star$ are linear functions of r, *i.e.* that they can be written on the form

$$x^\star = Mr, \qquad u^\star = Nr$$

and that the optimal controller for the tracking problem is on the form

$$u(t) = l_r r - Lx(t) \tag{6}$$

and derive an expression for l_r. (3p)

(c) In the lectures, we also used a controller on the form

$$u(t) = l_c r - Lx(t) \tag{7}$$

but we computed l_c in another way. We simply said that we first compute the state feedback for the standard LQ control problem (3-4) and then adjust l_c so that the stationary gain of the closed-loop system from r to y equals one. Are the two controllers the same?

If you are unable to work out the general solution, it is enough to consider the special case of a scalar system on the form

$$\begin{aligned} x(t) &= ax(t) + u(t) \\ z(t) &= cx(t) \\ y(t) &= x(t) \end{aligned}$$

and compare the expressions for l_r and l_c that the two approaches give. (3p)