EL2520
Control Theory and Practice

Lecture 13:
Model predictive control

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Learning aims

After this lecture you should be able to

- express finite-horizon constrained LQR problems as quadratic programs
- explain the basic idea of model predictive control
  - apply constrained optimal control in receding-horizon fashion
- enforce integral action in an MPC controller
- explain the issue of infeasibility and know how to circumvent it
- limit computational requirements of MPC by limiting control horizon

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Last lecture: finite-horizon LQR

Find control sequence

$$U = \{u_0, \ldots, u_{N-1}\}$$

that minimizes the quadratic cost function

$$J(U) = \sum_{k=0}^{N-1} (x_k^T Q_1 x_k + u_k^T Q_2 u_k) + x_N^T Q_f x_N$$

for given state cost, control cost, and final cost matrices

$$Q = Q^T \geq 0, \quad R = R^T > 0, \quad Q_f = Q_f^T \geq 0$$

N is called the **horizon** of the problem. Note the final state cost.

Optimal solution via quadratic minimization or dynamic programming.

Last lecture: finite-horizon LQR

Dynamic programming solution

1. set $P_N = Q_f$

2. for $t = N, N-1, \ldots, 1$

$$P_{t-1} := Q_1 + A^T P_t A - A^T P_t B (Q_2 + B^T P_t B)^{-1} B^T P_t A$$

3. for $t = 0, 1, \ldots, N-1$

$$L_t := (Q_2 + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$
$$u_t^\star = -L_t x_t$$

Note: optimal control is a linear function of the state

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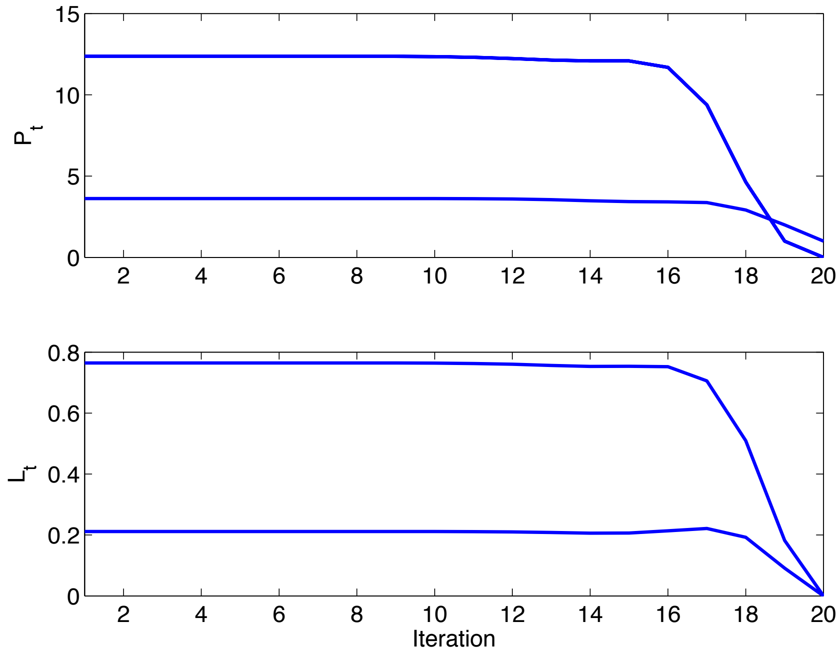
Example

Same system as earlier. Investigate how elements of P and L converge



Rapid convergence to stationarity as t drops below horizon N!

Last lecture: receding horizon LQR

Consider the cost function

$$J_k(u_k, \ldots, u_{k+K-1}) = \sum_{t=k}^{k+K-1} (x_t^T Q_1 x_t + u_t^T Q_2 u_t) + x_{k+K}^T Q_f x_{k+K}$$

Here, K is called the **horizon**, and if

$$(u_k^\star, \ldots, u_{k+K-1}^\star)$$

minimizes J_k, then $u_k^\star$ is called **K-step optimal receding horizon control**

Receding-horizon control:

- at time k, find input sequence that minimizes K-step ahead LQR cost (starting at time k)
- then apply only the first element of the input sequence

Last lecture: receding horizon LQR

Two ways to ensure closed-loop stability:

1. Use terminal cost matrix $Q_f = P$ where

   $$P = Q_1 + A^T PA - A^T PB(Q_2 + B^T PB)^{-1} B^T PA$$

   (i.e. P solves the discrete-time ARE) ensures stability.

   Why? Receding horizon-control is then (independent of t)

   $$u_t = -Lx_t \qquad L = (Q_2 + B^T PB)^{-1} B^T PA$$

   and the associated closed-loop system is stable
   (if basic observability and controllability conditions are met)

2. Use longer horizon, so that control approaches stationary optimal

Today: constrained predictive control

Finite-horizon LQG with hard constraints on u and y:

$$
\begin{array}{ll}
\text{minimize} & J(U) = \sum_{k=0}^{N-1} (x_k^T Q_1 x_k + u_k^T Q_2 u_k) + x_N^T Q_f x_N \\
\text{subject to} & u_{\min} \leq u_k \quad \leq u_{\max}, \quad k = 0, \dots, N-1 \\
& y_{\min} \leq C x_k \leq y_{\max}, \quad k = 1, \dots, N \\
& x_{k+1} = A x_k + B u_k
\end{array}
$$

Can be simplified by eliminating $\{x_1, \dots, x_N\}$ (as in last lecture)

– results in a quadratic programming problem in $\{u_0, \dots, u_{N-1}\}$

Quadratic programming (QP)

Minimizing a quadratic objective function subject to linear constraints

$$\begin{array}{ll} \text{minimize} & u^T P u + 2q^T u + r \\ \text{subject to} & Au \leq b \end{array}$$

Any u satsifying $Au \leq b$ is said to be **feasible.**

- clearly, not all quadratic programs are feasible
  (depends on A, b; more about this later…)

"Easy" to solve when objective function is **convex** (P positive semidefinite)

- optimal solution found in polynomial time
- commercial solvers deal with 1000's of variables in a few seconds

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Quadratic programming tricks

Example. The double inequality $u_{\min} \leq u \leq u_{\max}$ can be written as

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} u \leq \begin{bmatrix} u_{\max} \\ -u_{\min} \end{bmatrix}$$

Example. The equality $u = u_{\text{tgt}}$ can be written as $u_{\text{tgt}} \leq u \leq u_{\text{tgt}}$, hence

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} u \leq \begin{bmatrix} u_{\text{tgt}} \\ -u_{\text{tgt}} \end{bmatrix}$$

Constrained control via QP

$$\begin{array}{ll} \text{minimize} & J(U) = \sum_{k=0}^{N-1}(x_k^T Q_1 x_k + u_k^T Q_2 u_k) + x_N^T Q_f x_N \\ \text{subject to} & u_{\min} \leq u_k \quad \leq u_{\max}, \quad k = 0, \ldots, N-1 \\ & y_{\min} \leq C x_k \leq y_{\max}, \quad k = 1, \ldots, N \\ & x_{k+1} = A x_k + B u_k \end{array}$$

As in last lecture, introducing $X = (x_0, \ldots, x_N),\ U = (u_0, \ldots, u_{N-1})$,

$$X = GU + Hx_0$$

and the objective function can be written as

$$J(U) = U^T P_{LQ} U + 2 q_{LQ}^T U + r_{LQ}$$

Convex since $Q_2 \succ 0$ (see last lecture for precise expressions)

What about the constraints?

Predictive control with constraints

Similarly, the constraints $y_{\min} \leq y_k \leq y_{\max}, \quad k = 0, \dots, N$ can be written as

$$Y \geq y_{\min}\mathbf{1}, \quad Y \leq y_{\max}\mathbf{1}$$

where $Y = (y_0, \dots, y_N)$. Introducing

$$\overline{C} = \begin{bmatrix} C & 0 & \dots & 0 \\ 0 & C & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & C \end{bmatrix}$$

we can re-write these inequalities in terms of U via

$$Y \geq y_{\min}\mathbf{1} \Leftrightarrow \overline{C}(GU + Hx_0) \geq y_{\min}\mathbf{1} \Leftrightarrow \underbrace{\overline{C}G}_{A_{\underline{Y}}} U \geq \underbrace{y_{\min}\mathbf{1} - \overline{C}Hx_0}_{b_{\underline{Y}}}$$

$$Y \leq y_{\max}\mathbf{1} \Leftrightarrow \overline{C}(GU + Hx_0) \leq y_{\max}\mathbf{1} \Leftrightarrow \underbrace{\overline{C}G}_{A_{\overline{Y}}} U \leq \underbrace{y_{\max}\mathbf{1} - \overline{C}Hx_0}_{b_{\overline{Y}}}$$

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Predictive control with constraints

Hence, the constrained predictive control problem can be cast as a QP

$$
\begin{array}{ll}
\text{minimize} & U^T P_{LQ} U + 2 q_{LQ}^T U + r_{LQ} \\
\text{subject to} & \begin{bmatrix} A_{\overline{Y}} \\ -A_{\underline{Y}} \\ I \\ -I \end{bmatrix} U \leq \begin{bmatrix} b_{\overline{Y}} \\ -b_{\underline{Y}} \\ u_{\max}\mathbf{1} \\ -u_{\min}\mathbf{1} \end{bmatrix}
\end{array}
$$

Solution gives optimal finite-horizon control subject to constraints

Model predictive control:

– apply constrained optimal control in receding horizon fashion

Model predictive control algorithm

1. Given state at time t compute ("predict") future states

$$x_{t+k}, \qquad k = 0, 1, \dots, N$$

   as function of future control inputs

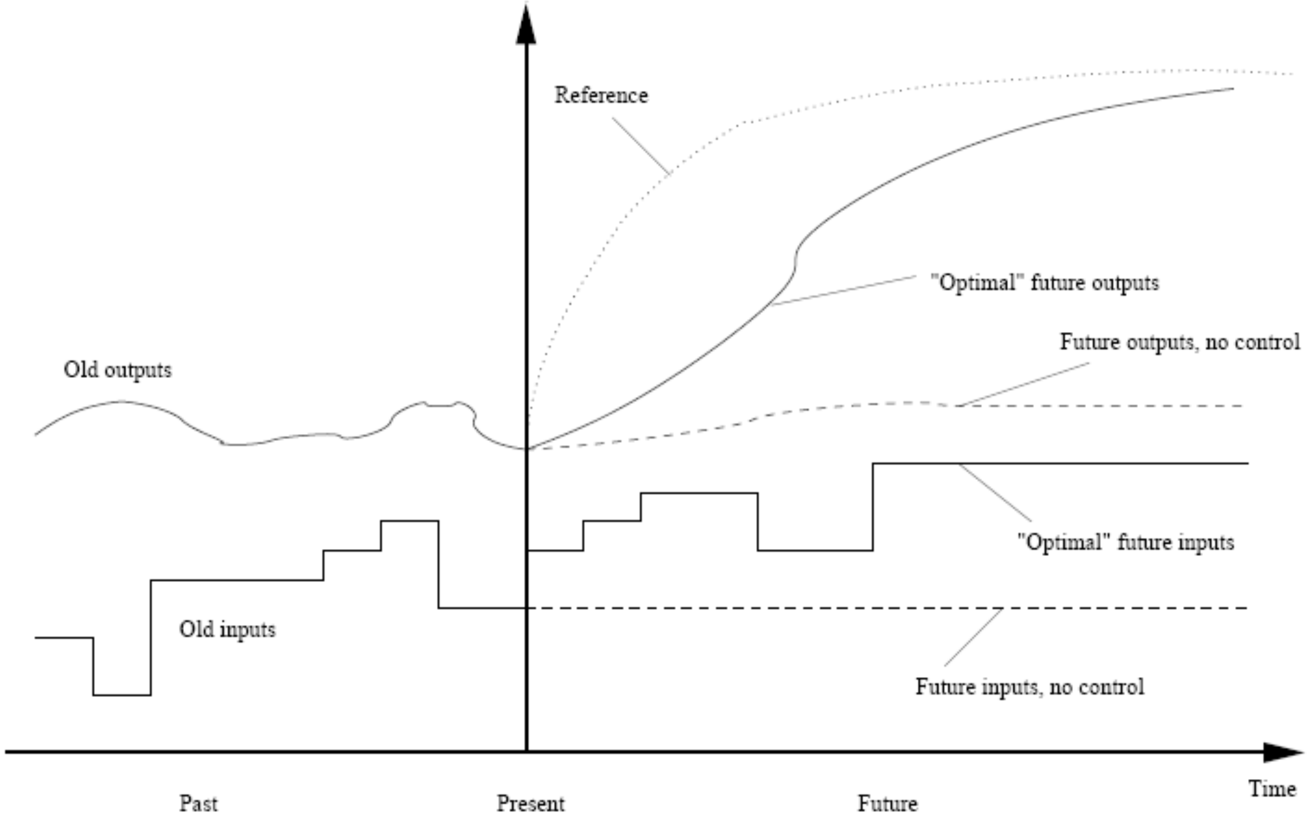
$$u_{t+k}, \qquad k = 0, 1, \dots, N-1$$

2. Find "optimal" input by minimizing constrained cost function
   - a quadratic program, efficiently solved

3. Implement u(t)

4. A next sample (t+1), return to 1.

MPC in pictures



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Example: the DC servo

Discrete-time model (sampling time 0.05 sec)

$$A = \begin{bmatrix} 1 & 0.139 \\ 0 & 0.861 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0107 \\ 0.139 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Constrained input voltage

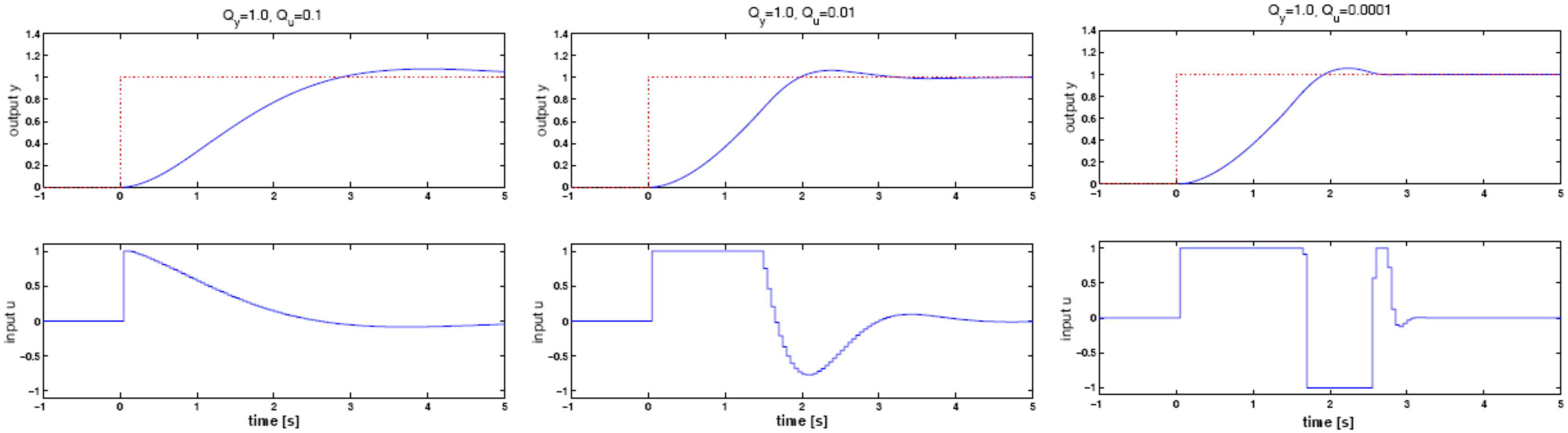
$$-1 \leq u \leq 1$$

Constrained position

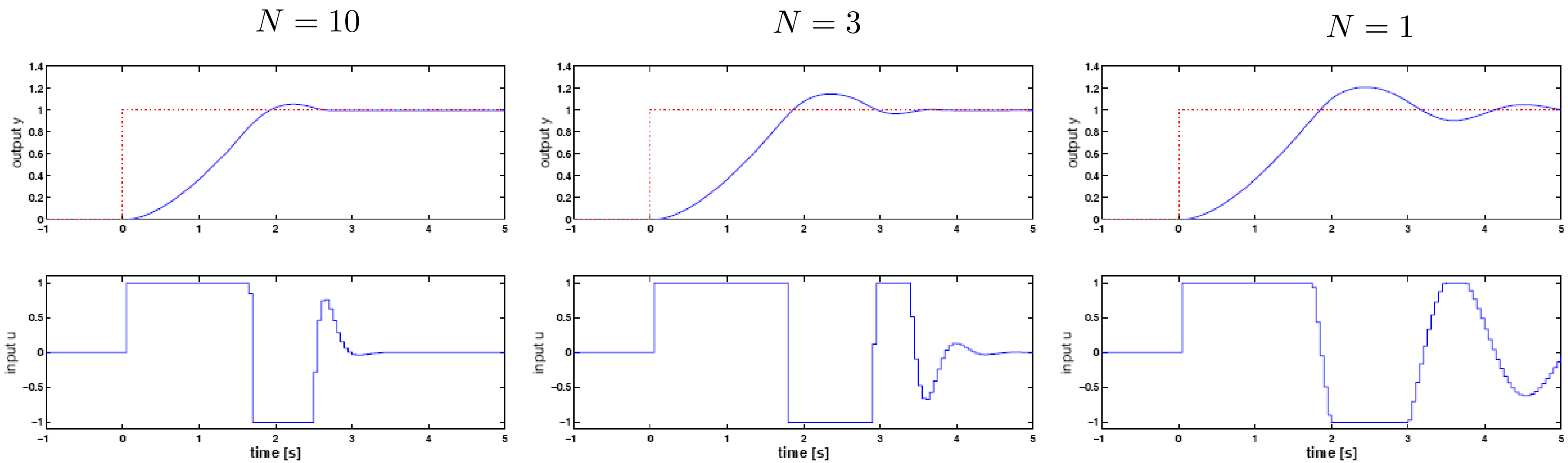
$$y_{\text{min}} \leq y_k \leq y_{\text{max}}$$

Impact of state and control weights

Prediction horizon N=10.



Impact of horizon



Too short horizon➔inaccurate predictions➔poor performance

Reference tracking

Would like z to track a reference sequence $\{r_1, \ldots, r_N\}$, i.e. to keep

$$\sum_{k=0}^{N-1} \big((z_k - r_k)^T Q_1 (z_k - r_k) + u_k^T Q_2 u_k\big) + (z_N - r_N)^T Q_f (z_N - r_N)$$

small.

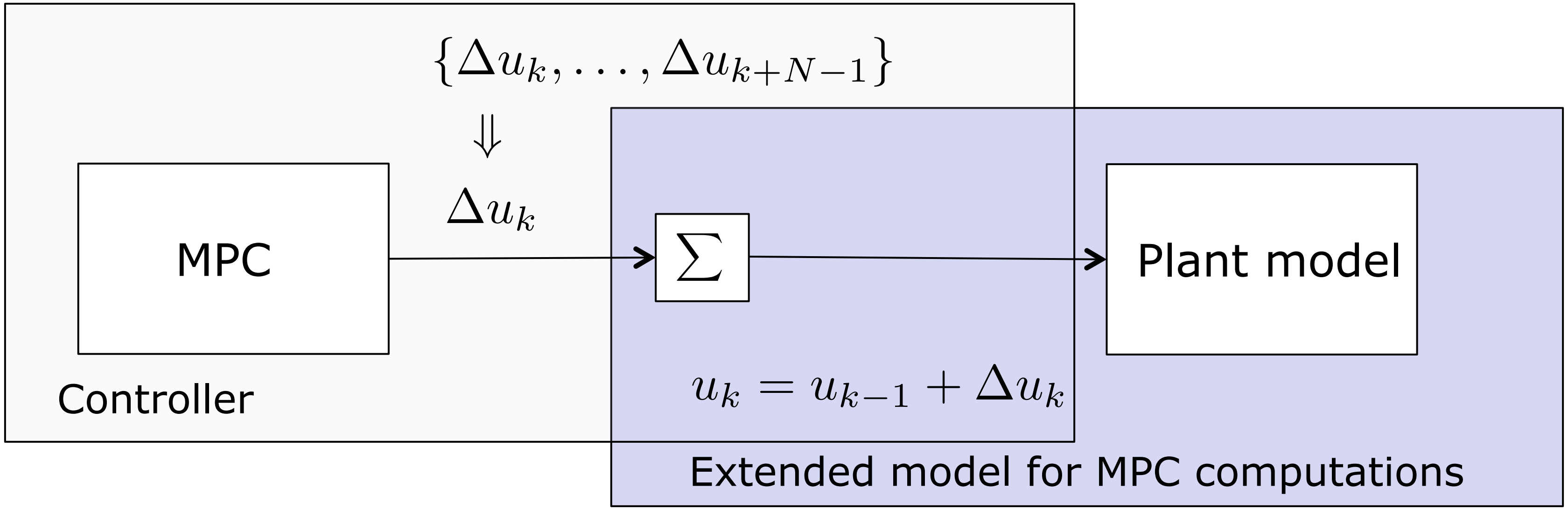
Problem: making $z_k = r_k$ typically requires $u_k \neq 0$

- a trade-off between zero tracking errors and using zero control
- often results in steady-state tracking error

Including integral action

Integral action often included by a change in free variables

- Use $\Delta u_i = u_i - u_{i-1}$ as variables in the optimization
- Actual input obtained by summing up MPC outputs



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Including integral action cont'd

Form augmented model with state $\bar{x}_k = \begin{pmatrix} x_k & u_{k-1} \end{pmatrix}$ and input Δu_k:

$$\begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta u_k$$

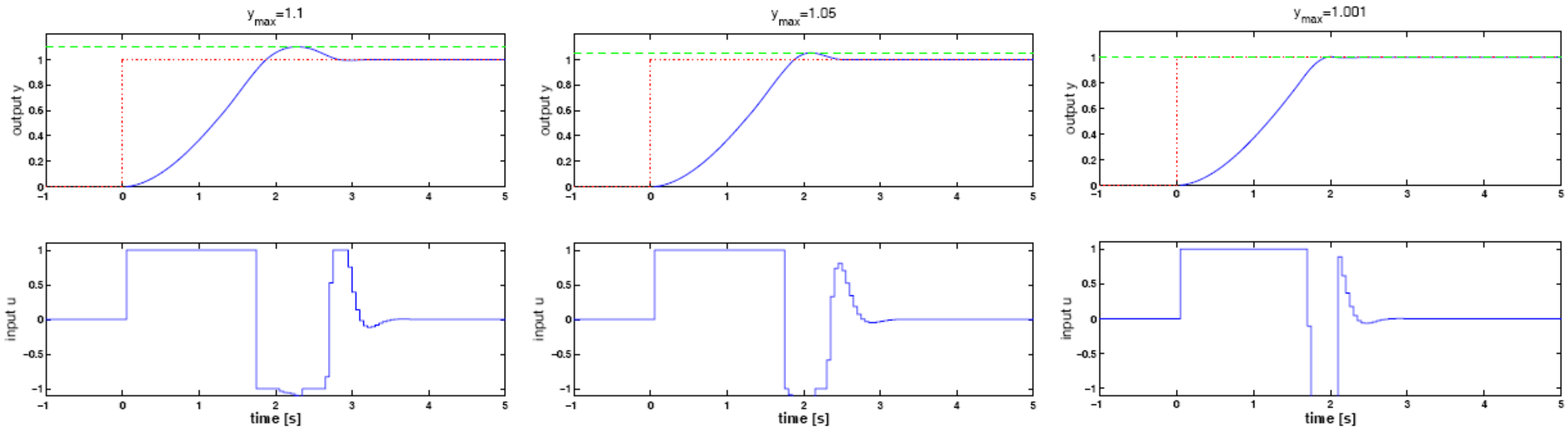
$$y_k = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix}$$

Consider finite-horizon cost

$$\sum_{k=0}^{N-1} \left((z_k - r_k)^T Q_1 (z_k - r_k) + \Delta u_k^T Q_2 \Delta u_k \right) + (z_N - r_N)^T Q_f (z_N - r_N)$$

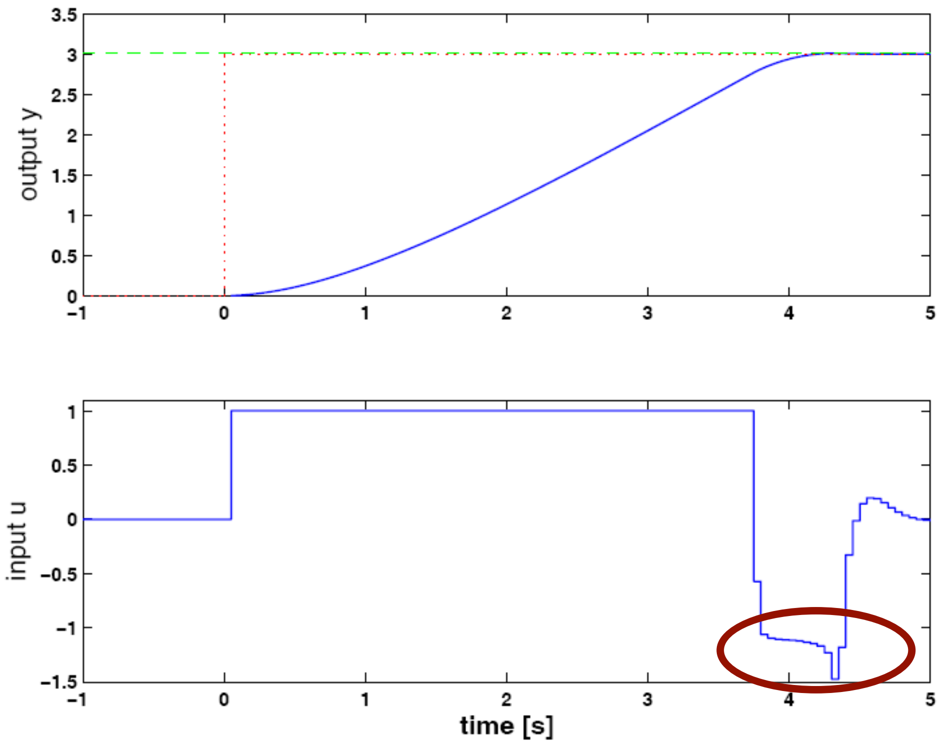
(easy to add penalty also to u_k …)

Tracking with output constraints



Infeasibility

What happens when there is no solution to the QP?



Not clear what control to apply!

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Ensuring feasibility

One way to ensure feasibility:

- introduce slack variables $s_{ck} \geq 0$
- "soften" constraints

$$u_k \leq u_{\max} \Rightarrow u_k \leq u_{\max} + s_{ck}$$

- add term in quadratic programming objective to minimize slacks

$$\underset{U}{\text{minimize}} \quad U^T P_{LQ} U + 2q_{LQ}^T U + r_{LQ}$$

$$\Downarrow$$

$$\underset{U,S}{\text{minimize}} \quad U^T P_{LQ} U + 2q_{LQ}^T U + r_{LQ} + \kappa S^T S$$

Notes:

- still QP, but more variables; can also use penalty κS (also QP)
- better to soften already soft constraints (e.g. output constraints)

Limited horizon control

The longer horizon, the more variables in the QP.

Idea: limit the number of variables by

- predicting (and constraining) the system over horizon of length N
- only optimizing the first $N_u \leq N$ control actions

(called input and output horizons, or control and prediction horizons)

Solutions depend on what we assume about u beyond control horizon.

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Limited horizon control I

Simple solution: control is held constant beyond control horizon

$$\{u_0, \ldots, u_{N_u-1}\} \text{ are optimized }, \quad u_{N_u} = \cdots = u_{N-1} = u_{N_u-1}$$

Can use the same QP formulation as before and add linear equalities

$$u_k = u_{N_u-1} \qquad k = N_u, N_{u+1}, \ldots, N$$

Can re-write as a QP on standard form (using techniques form earlier)

Modern solvers automatically remove these variables from QP.

Limited horizon control II

Alternative approach: use inner-loop feedback

$$u_k = -Lx_k, \qquad k = N_u, N_{u+1}, \dots, N$$

Prediction model becomes

$$x_{k+1} = Ax_k + Bu_k \qquad k = 0, 1, \dots, N_{u-1}$$
$$x_{N_u+t} = (A - BL)^t x_{N_u} \qquad t = 0, 1, \dots, N - N_u - 1$$

With $X = (x_0, \dots, x_N)$, $\bar{U} = (u_0, \dots, u_{N_u-1})$, we can write

$$X = \overline{GU} + \overline{H}x_0$$

So finite-horizon constrained optimal control can be found via QP.

Note: beyond control horizon, $u_{\min} \leq u_k \leq u_{\max} \Leftrightarrow u_{\min} \leq -Lx_k \leq u_{\max}$ (constraints on control translates into constraints on predicted states)

MPC controller tuning

MPC has a large number of “tuning” parameters.

The prediction model:

- we need to decide sampling interval
  (rule of thumb: sample 10 times desired closed-loop bandwidth)
- obtain discrete-time state-space model

Finite-horizon optimal control:

- set prediction horizon
  (rule of thumb: equal to closed-loop rise time; could be smaller)
- decide weight matrices (as for continuous-time LQG)
- decide final state penalty
  (guideline: stationary Riccati solution for given weight matrices)

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MPC controller tuning

Finite-horizon optimal control, advanced:

- control horizon (try to set small, rule-of-thumb: use 1-10)
- inner-loop control
  (guideline: stationary LQR controller for given weight matrices)

Constraints and feasibility

- specify control and state constraints (problem dependent)
- introduce slacks to "soften" constraints
- choose constraint penalty (large value on kappa)

Integral action (almost always a good idea to include).

Advanced issues: stability

Receding horizon control might yield unstable closed-loop

Stability can be guaranteed:

- for infinite-horizon unconstrained case (this is LQR)
- for finite-horizon unconstrained case
  - if final state is penalized correctly
  - if final state is enforced to lie in a given set
- for constrained finite-horizon
  - if final state enforced to lie in a sufficiently small set **and**
  - initial QP (solved at time zero) is feasible

Hard to verify for sure in advance…

Advanced issues: robustness

Consider the unconstrained quadratic program

$$\text{minimize} \quad u^T Q u + 2q^T u$$

has optimal solution $u = -Q^{-1} q$

In the MPC setting, Q and q depend on the system model (matrices A, B, C), weights Q_1, Q_2, and also horizons.

Solution is sensitive to uncertainties if Q is ill-conditioned

- Try scaling inputs and outputs in the model
- Modify weight matrices Q_1 and Q_2
- Almost always a good idea to include integral action

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Advanced issues: observers

MPC, as presented here, assumes full state feedback.

In many cases, we will need to use an observer,

- to reconstruct states, and/or
- to filter out noise

Limited theory, but separation principle holds in some cases.

Suggests guideline

- design observer as for (unconstrained, infinite-horizon) LQG
- use estimated state in MPC calculations as if it was true state

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Get a feel for MPC!

Files for setting up and simulating a problem is on home page.

Do Computer Exercise 5 (download from home page)!

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Summary

Model predictive control (MPC)

- Can handle state and control constraints
- Predictive control computed via quadratic programming

Many parameters and their influence on the control

- System model, weights, horizons, constraints, …

Advanced issues:

- The need for a state observer
- Different prediction and control horizons
- Feasibility and slacks to "soften" constraints
- Stability and the terminal weight
- Integral action

Do computer exercise 5 to get a feel for MPC!