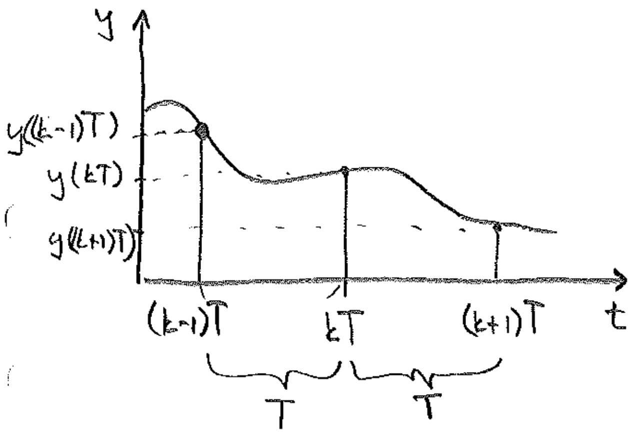
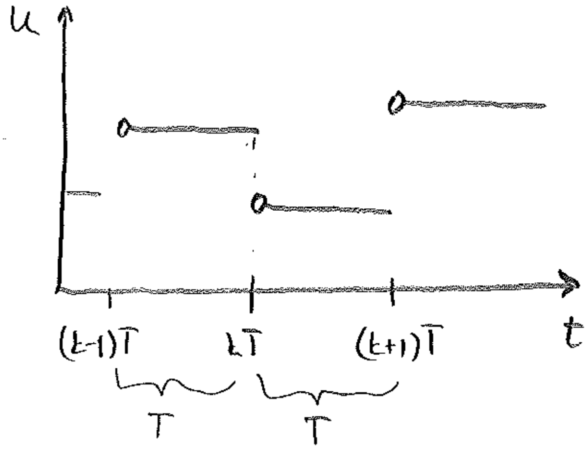
Theory

Discrete time:

Consider the system at discrete points in time.



Assumes u piecewise constant



Notation: $u_k = u(kT)$ $\qquad u_{k+1} = u((k+1)T)$

$y_k = y(kT)$ $\qquad \ldots$

$x_k = x(kT)$

$(x_k)_1 = x_1(kT)$ i.e. $(x_k)_1$ $\qquad \begin{bmatrix} x_1(kT) \\ x_2(kT) \\ x_3(kT) \end{bmatrix}$

$(x_k)_3$

Difference equations

$$x_{k+1} = F x_k + G u_k$$
$$y_k = H x_k$$

Sampling: Continous → Discrete

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Sample with interval T ⇒

$$x_{k+1} = F x_k + G u_k$$
$$y_k = C x_k$$

$$F = e^{AT}, \quad G = \int_0^T e^{At} B\, dt, \quad H = C$$

Matrix exponential:

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}$$

alternative

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

Stability: λ_i = eigenvalue of F

stable if $|\lambda_i| < 1 \ \forall i$

Observable: Same as for cont. sys.

observable if $O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$ has full rank

Controllable:

Controllable if $S = [G \ FG \ F^{n-1}G]$ has full rank.

Model Predictive Control (MPC):

We can predict future outputs as functions of future inputs.

Ex:

$$\hat{y}_{k+1} = Hx_{k+1} = H(Fx_k + Gu_k) = HFx_k + HGu_k$$

Idea: minimize cost J involving future outputs.

1) Solve $\min_u J(u, \hat{y})$ $\qquad u = \begin{pmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+N} \end{pmatrix}$

s.t. $x_{k+1} = Fx_k + Gu_k$ $\qquad \hat{y} = \begin{pmatrix} \hat{y}_{k+1} \\ \vdots \\ y_{k+M} \end{pmatrix}$

$y_k = Hx_k$

$\Rightarrow$ optimal control sequence u_k

2) apply first control u_k and go to 1)

<u>Note:</u> Can include hard constraints such as $|u| < 1$ or $|y| < 1$

MPC is computationally expensive.

4.1

The cont. system

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{(1 \quad 0)}_{C} x(t)$$

is sampled with interval T. Determine the corresponding discrete time system. Is the sampled system observable?

i) find F, G & H.

$$F = e^{AT} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \mathcal{L}^{-1}\left\{\begin{pmatrix} s & \omega \\ -\omega & s \end{pmatrix}^{-1}\right\} =$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\begin{pmatrix} s & -\omega \\ \omega & s \end{pmatrix}\right\} = \begin{pmatrix} \cos(\omega T) & \sin(\omega T) \\ -\sin(\omega T & \cos(\omega T) \end{pmatrix}$$

$$G = \int_0^T e^{At} B\, dt = \int_0^T \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} dt =$$

$$= \int_0^T \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{pmatrix} dt = \left[\frac{1}{\omega}\begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \end{pmatrix}\right]_0^T = \frac{1}{\omega}\begin{pmatrix} \sin(\omega T \\ \cos(\omega T) - 1 \end{pmatrix}$$

$$H = C$$

The system is thus

$$x_{k+1} = \begin{pmatrix} \cos(\omega T) & \sin(\omega T) \\ -\sin(\omega T) & \cos(\omega T) \end{pmatrix} x_k + \frac{1}{\omega} \begin{pmatrix} \sin(\omega T) \\ \cos(\omega T) - 1 \end{pmatrix} u_k$$

$$y_k = \begin{pmatrix} 1 & 0 \end{pmatrix} x_k$$

ii) Observable?

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ \cos(\omega T) & \sin(\omega T) \end{pmatrix}$$

Full rank if $\sin(\omega T) \neq 0 \Rightarrow$

$\omega T \neq n \cdot \pi \quad n \in \mathbb{Z}$

$T \neq \frac{n\pi}{\omega} \quad n \in \mathbb{Z} \Rightarrow$ observable.

16.2 Given the discrete time system

$$x_{k+1} = F x_k + G u_k \qquad F = \begin{pmatrix} 1,7 & 1 \\ -0,7 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0,9 \\ -0,6 \end{pmatrix}$$

$$y_k = H x_k \qquad H = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Find a control law which solves

$$\min_u J = \min_u \sum_{i=1}^{3} \underbrace{(r_{k+i} - \hat{y}_{k+i})^2}_{\text{predicted control error}} + 0,1 \underbrace{(u_{k+i-1})^2}_{\text{control effort}}$$

where $\hat{y}_{k+i}$ is the predicted output i steps ahead.

<u>Rewrite J with vectors:</u>

let $\vec{r} = \begin{pmatrix} r_{k+1} \\ r_{k+2} \\ r_{k+3} \end{pmatrix} \quad \vec{y} = \begin{pmatrix} \hat{y}_{k+1} \\ \hat{y}_{k+2} \\ \hat{y}_{k+3} \end{pmatrix} \quad \vec{u} = \begin{pmatrix} u_k \\ u_{k+1} \\ u_{k+2} \end{pmatrix}$

then $J = (\vec{r} - \vec{y})^T (\vec{r} - \vec{y}) + 0,1\, \vec{u}^T \vec{u}$

<u>Express J in terms of $\vec{u}$:</u>

$$\hat{y}_{k+1} = H x_{k+1} = H(F x_k + G u_k) = HF x_k + HG u_k$$

$$\hat{y}_{k+2} = H x_{k+2} = H(F x_{k+1} + G u_{k+1}) = H(F(F x_k + G u_k) + G u_{k+1}) =$$
$$= HF^2 x_k + HFG u_k + HG u_{k+1}$$

$$\hat{y}_{k+3} = \ldots = HF^3 x_k + HF^2 G u_k + HFG u_{k+1} + HG u_{k+2}$$

$$\Rightarrow \quad \vec{y} = \underbrace{\begin{pmatrix} HF \\ HF^2 \\ HF^3 \end{pmatrix}}_{P} x_k + \underbrace{\begin{bmatrix} HG & 0 & 0 \\ HFG & HG & 0 \\ HF^2G & HFG & HG \end{bmatrix}}_{S} \vec{u}$$

$$\vec{y} = Px_k + S\vec{u} \quad \Rightarrow$$

$$J = (\vec{r} - Px_k - S\vec{u})^T(\vec{r} - Px_k - S\vec{u}) \quad + \quad 0{,}1\, \vec{u}^T\vec{u}$$

Expand J:

$$J = \vec{r}^T\vec{r} - 2\vec{r}^T P x_k + x_k^T P^T P x_k - 2\vec{r}^T S\vec{u} + 2x_k^T P^T S\vec{u} + \vec{u}^T S^T S\vec{u} + 0{,}1\vec{u}^T\vec{u}$$

J Quadratic in $\vec{u}$ $\Rightarrow$ easy to minimize.

$$\frac{\partial}{\partial \vec{u}} J = -2S^T\vec{r} + 2S^T P x_k + 2S^T S\vec{u} + 0{,}2I\vec{u} = 0$$

$\Rightarrow$ the minimizing $\vec{u} = (S^T S + 0{,}1 I)^{-1} S^T(\vec{r} - Px_k)$

Inserting values for F, G, & H $\Rightarrow$

$$P = \begin{pmatrix} 1{,}7 & 1 \\ 2{,}2 & 1{,}7 \\ 2{,}5 & 2{,}2 \end{pmatrix} \qquad S = \begin{pmatrix} 0{,}9 & 0 & 0 \\ 0{,}93 & 0{,}9 & 0 \\ 0{,}95 & 0{,}93 & 0{,}9 \end{pmatrix}$$

We are only interested in u_k (remember that we will recompute $u_{k+1}, \ldots$ next iteration)

Doing the multiplications etc. $\Rightarrow$

$u_k = -(1{,}7 \quad 1{,}1)\, x_k + (0{,}9 \quad 0{,}1 \quad 0{,}01)\, \vec{r}$

We have u_k expressed as a state-feedback.
We want to express it in terms of the measurements we have access to. $(y_k, y_{k-1}, \ldots)$

Note that

$(x_k)_1 = y_k \qquad (x_k)_2 = (x_{k-1})_1 - 0{,}6\, u_{k-1} = y_{k-1} - 0{,}6\, u_{k-1}$

current measurement

last measurement

last applied control

The final control law becomes

$u_k = -1{,}7\, y_k - 1{,}1\, y_{k-1} + 0{,}64\, u_{k-1} + 0{,}9\, r_{k+1} + 0{,}1\, r_{k+2} + 0{,}01\, r_{k+3}$

<u>Note:</u> need to know future references.

16.4

A system with model

$$y_{k+1} = -y_k + 2u_k$$

is to be controlled by an MPC-controller

$$\min_u \sum_{i=k}^{k+N_p} y_i^2 + \sum_{i=k}^{k+N_p-1} u_i^2$$

$$\text{s.t. } |u| \leq 1$$

Let $N_p = 1$ and translate the problem into a quadratic programming problem on the form

$$\min_u u^T H u + h^T u \quad \text{s.t. } Lu \leq b$$

i.e. find H, h^T, L & b.

Expand J:

$$J = \sum_{i=k}^{k+1} y_i^2 + \sum_{i=k}^{k} u_i^2 = y_{k+1}^2 + y_k^2 + u_k^2$$

Use system model to get

$$J = (-y_k + 2u_k)^2 + y_k^2 + u_k^2 = 2y_k^2 - 4y_k u_k + 5u_k^2$$

Note that $\arg\min_{u_k} 2y_k^2 - 4y_k u_k + 5u_k^2 = \arg\min_{u_k} -4y_k u_k + 5u_k^2$

since $2y_k^2$ is independent of u_k

Hence we can instead solve

$$\min_{u_k} \; u_k^T \underbrace{5}_{H} u_k \underbrace{- 4 y_k}_{h^T} u_k$$

<u>The constraint:</u>

$$|u_k| \leq 1 \iff -1 \leq u_k \,, \quad u_k \leq 1 \iff -u_k \leq 1 \,, \quad u_k \leq 1$$

which can be expressed as

$$\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{L} u_k \leq \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b}$$

<u>Note:</u> The comparison is element wise.

$$H = 5 \qquad h^T = -4y_k \qquad L = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$