KTH Electrical Engineering

Exam in EG2080 Monte Carlo Methods in Engineering, 7 June 2013, 12:30–16:30, the seminar room

Instructions

The answer to each problem must begin on a new sheet, but answers to different parts of the same problem (a, b, c, etc.) can be written on the same sheet. The fields *Namn* (Name), *Blad nr* (Sheet number) and *Uppgift nr* (Problem number) must be filled out on every sheet.

Solutions should include sufficient detail that the argument and calculations can be easily followed.

The exam can yield 40 points in total. The examinee is guaranteed to pass if the score is at least 33 points. There will be a possibility to complement for passing the exam if the result is at least 31 points.

Allowed aids

The following aids are allowed in this exam:

- Calculator without information relevant to the course.
- Formulae sheet.

Problem 1 (8 p)

Y is a two-state random variable with the probability density function

$$f_Y(x) = \begin{cases} 0.1 & x = 0, \\ 0.9 & x = 1, \\ 0 & \text{all other } x. \end{cases}$$

Table 1 Random numbers from a $U(0, 1)$ distribution

0.44	0.77	0.19	0.45	0.71
0.38	0.80	0.49	0.65	0.75

a) (2 p) Apply the inverse transform method to generate 10 values of Y. This problem should be solved using all the random numbers in table 1.

b) (2 p) Apply complementary random numbers to generate 10 values of Y. This problem should be solved using the first row of the random numbers in table 1.

c) (2 p) Apply dagger sampling to generate 10 values of Y. This problem should be solved using as many of the random numbers in table 1 as needed.

d) (2 p) Compare the results of part a–c. Which sequence of random numbers is closest to the actual probability distribution? Is this a coincidence or is there an explanation to the result?

Problem 2 (8 p)

The following results have been obtained from a Monte Carlo simulation using simple sampling:

$$\sum_{i=1}^{1\,000} x_i = 1\,970, \quad \sum_{i=1}^{1\,000} x_i^2 = 4\,872.$$

a) (2 p) Calculate the estimated expectation value of X.

b) (2 p) Calculate the estimated variance of X.

c) (4 p) Calculate a 99% confidence interval for the expectation value of X. It can be assumed that M_X is normally distributed.

Problem 3 (8 p)

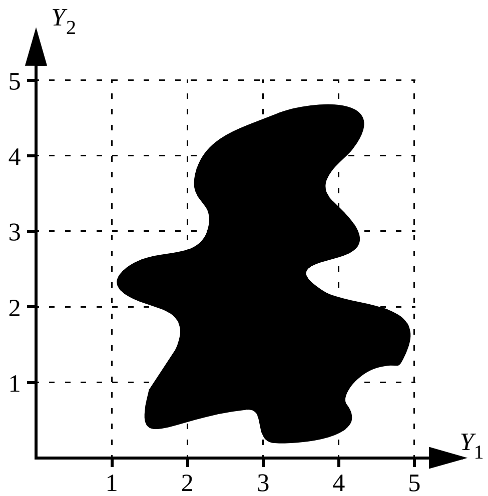
Table 2 lists the possible states of a simplified model of a system with two outputs. Suggest an importance sampling function for this system.

Table 2 Scenarios for the system in table 3.

Input, y	Probability $f_Y(y)$	Outputs from the simplified model z_1	z_2
1	0.1	150	0
2	0.2	275	0
3	0.4	475	0
4	0.2	700	0.5
5	0.1	1 000	1

Problem 4 (8 p)

The area of the shape in the figure below is to be estimated using stratified sampling.



a) (5 p) Use a strata tree to define appropriate strata.

b) (3 p) Calculate the stratum weights for the strata that were defined in part a.

Problem 5 (8 p)

Monte Carlo simulation has been used to compare the two systems $X_1 = g_1(Y)$ and $X_2 = g_2(Y)$. Some results of the simulation are shown in table 3. Calculate the estimated difference between $E[X_1]$ and $E[X_2]$. If the table does not contain all the information you need to solve the problem, then you may assume an arbitrary value for the missing information.

Table 3 Results from the Monte Carlo simulation in problem 5.

Stratum, h	Stratum weight, ω_h	Samples per stratum, n_h	Results from system 1 $\sum_{i=1}^{n_h} g_1(y_{h,i})$	Results from system 2 $\sum_{i=1}^{n_h} g_2(y_{h,i})$
1	0.5	300	900 000	750 000
2	0.3	300	1 470 000	1 260 000
3	0.2	400	2 250 000	1 850 000

Problem 1

a) The inverse distribution function is given by

$$F_Y^{-1} = \begin{cases} 0 & \text{if } 0 \le x \le 0.1, \\ 1 & \text{if } 0.1 < x \le 1. \end{cases}$$

It is then easy to see that all the random numbers in table 1 will be transformed into the value 1, i.e., $y_i = 1$, $i = 1, \ldots, 10$.

b) We now apply the inverse transform in the same manner as in the previous problem, but we now use five values from table 1, u_i, $i = 1, \ldots, 5$, whereas the remaining five values to be transformed are given by $1 - u_i$, $i = 1, \ldots, 5$. In practice this means that the first five values are the same as in part a, and for the remaining five values we should look in the first row of the table for entries larger than 0.9; however, where are no such entries, which means that we will get $y_i = 1$, $i = 1, \ldots, 10$, in this case as well.

c) The dagger cycle length is given by $S = \text{floor}(1/p) = \{p = 0.1\} = 10$; thus, we will only need to generate one dagger cycle (which means that we are only going to need one random number). The first value in table 1 is 0.44, which points to the fifth interval, i.e., the condition $p(j-1) \le x < p \cdot j$ in the dagger transform is fulfilled for $j = 5$. The resulting sequence is then that $y_i = 1$, $i = 1, \ldots, 4, 6, \ldots, 10$ and $y_5 = 0$.

d) With the inverse transform method and complementary random numbers, the value 1 does not appear in the generated sequence, whereas for dagger sampling it appears in one out of scenarios, which is exactly according to the actual probability distribution of Y. The result of dagger sampling is not a coincidence, as there is no rest interval in the dagger transform and we are generating a complete dagger cycle! For other pseudorandom numbers, the inverse transform method and complementary random numbers could have generated one or more results where $y_i = 0$. However, the probability of getting the results in part a and b are not insignificant ($0.9^{10} \approx 35\%$ for the inverse transform method and $0.8^5 \approx 33\%$ for complementary random numbers).

Problem 2

a) $m_X = \frac{1}{1\,000}\sum_{i=1}^{1\,000} x_i = 1.97.$

b) $s_X^2 = \frac{1}{1\,000}\left(\sum_{i=1}^{1\,000}(x_i - m_X)^2\right) = \frac{1}{1\,000}\left(\sum_{i=1}^{1\,000} x_i^2\right) - m_X^2 \approx 0.99.$

c) $\delta = \frac{t_\alpha \cdot s_X}{\sqrt{n}} \approx \{t_{0.99} = 2.5758\} \approx 0.022 \Rightarrow 1.97 \pm 0.08$ is a 99% confidence interval for $E[X]$.

Problem 3

The expectation values of the simplified models are

$$\mu_{Z1} = \sum_{\psi=1}^{5} f_Y(\psi) z_1(\psi) = 410, \quad \mu_{Z2} = \sum_{\psi=1}^{5} f_Y(\psi) z_2(\psi) = 0.14.$$

The optimal importance function for each output is then given by

$$f_{Z1}(\psi) = \frac{f_Y(\psi) z_1(\psi)}{\mu_{Z1}} = \begin{cases} 0.03 & \psi = 1, \\ 0.11 & \psi = 2, \\ 0.38 & \psi = 3, \\ 0.28 & \psi = 4, \\ 0.20 & \psi = 5, \\ 0 & \text{all other } \psi, \end{cases}$$

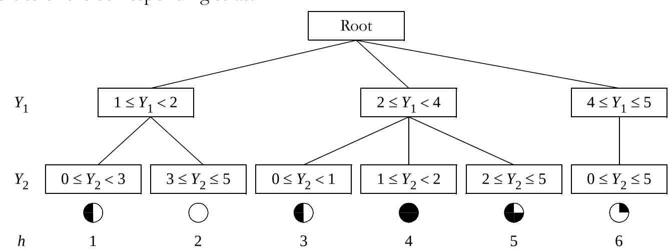
$$f_{Z2}(\psi) = \frac{f_Y(\psi) z_2(\psi)}{\mu_{Z2}} = \begin{cases} 0.5 & \psi = 4, \\ 0.5 & \psi = 5, \\ 0 & \text{all other } \psi. \end{cases}$$

A reasonable compromise of these two importance sampling functions is to use the average, i.e.,

$$f_Z(\psi) = \frac{f_{Z1}(\psi) + f_{Z2}(\psi)}{2} = \begin{cases} 0.015 & \psi = 1, \\ 0.055 & \psi = 2, \\ 0.190 & \psi = 3, \\ 0.390 & \psi = 4, \\ 0.350 & \psi = 5, \\ 0 & \text{all other } \psi. \end{cases}$$

Problem 4

a) To determine the shape of the area we can investigate the share of points within the rectangle $1 \le Y_1 \le 5$ and $0 \le Y_2 \le 5$. The idea is to separate areas which are either clearly inside or clearly outside the shape from areas which are both inside and outside the shape. One possible strata tree according to this principle is shown below. The circles below each branch indicates the expected properties of the corresponding stratum..



b) Both inputs should be uniformly distributed; hence, the stratum weight will be equal to the size of stratum divided by the size of the rectangle which comprises the sample space. Thus, we get the following stratum weights:

$\omega_1 = 1 \cdot 3/20 = 0.15, \quad \omega_2 = 1 \cdot 2/20 = 0.1, \quad \omega_3 = 2 \cdot 1/20 = 0.1,$

$$\omega_4 = 2 \cdot 1/20 = 0.1, \qquad \omega_2 = 2 \cdot 3/20 = 0.3, \qquad \omega_3 = 1 \cdot 5/20 = 0.25.$$

Problem 5

The information is sufficient. We start by calculating the estimated difference between the two systems for each stratum:

$$m_{(X1-X2)h} = \frac{1}{n_h} \sum_{i=1}^{h} (g_1(y_{h,i}) - g_2(y_{h,i})) = \begin{cases} 500 & h = 1, \\ 700 & h = 2, \\ 1\,000 & h = 3. \end{cases}$$

Then we calculate the estimated difference between the two systems for the entire population:

$$m_{(X1-X2)} = \sum_{h=1}^{3} \omega_h m_{(X1-X2)h} = 660.$$