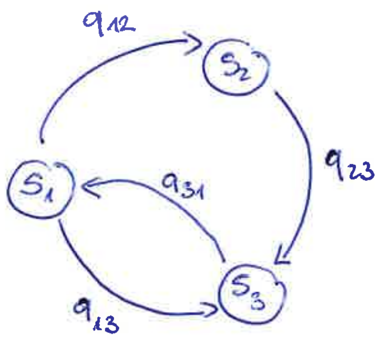
EP 2200 Queuing Theory, June 3, 2013

1)

a)



Conditions for stationary solution:

- finite number of states
- irreducible

Both are needed for stability!

b)

$$\underline{Q} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Matrix equation of the stationary solution:

$$\underline{p}\,\underline{Q} = \underline{0}$$

Stationary state distribution from matrix equation, or from balance eq:

$$\left.\begin{array}{l} p_1 1 = p_2 1 \\ p_2 1 + p_1 1 = p_3 1 \\ p_1 + p_2 + p_3 = 1 \end{array}\right\} [p_1, p_2, p_3] = [\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{2}{4}]$$

c) Holding time in S_1: Exponential with $q_1 = \sum_{1 \neq j} q_{1j} = 2$

$$P(\tau > 2) = 1 - F(2) = e^{-q_1 \tau} = e^{-4} = 0.018$$

d) Reparation time: $Exp(q_{31})$, $\bar{\tau}_3 = \frac{1}{q_{31}} = 1$ time unit

Average production: $\bar{N} = 100 p_1 + 40 p_2 + 0 p_3 = 35$

2)

a) M / M / 15 / ∞ , $\lambda = 1/\text{sec}$, $\mu = \frac{1}{12}/\text{sec}$ $(\bar{x} = 12 \text{ sec})$, $m = 15$

b) Offered load $A = \lambda \bar{x} = 12 = 8$

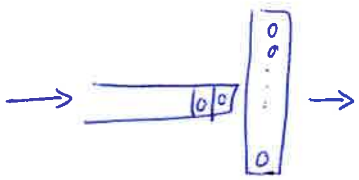
Utilization $= \frac{A}{m} = \frac{12}{15} = \frac{4}{5} = 0.8$

The system is stable, since utilization < 1.

c)
$$P(\text{wait}) = \frac{m \cdot P(\text{block})}{m - A(1-P(\text{block}))} = 0.317 \quad \Big| \; P(\text{block}) = 0{,}085$$

$$T = x + W = x + \frac{1}{m\mu - \lambda} P(\text{wait}) = 12 + \frac{1}{15 \cdot \frac{1}{12} - 1} 0.317 = 13.26 \text{ sec}$$

d)



$Y_i \mathrel{\hat=}$ time between service

$Y_i \sim Exp(m\mu) = Exp(\frac{5}{4})$

Number of services ~~wait~~ to wait: 3

$W = Y_1 + Y_2 + Y_3$

$$L(f_W(t)) = L(f_Y(t))^3 = \left(\frac{m\mu}{s + m\mu}\right)^3$$

Time domain: sum of 3 exp distributed r.v. ⇒ Erlang distribution:

$$f_W(t) = \frac{(m\mu)^3 t^2}{2} e^{-m\mu t}$$

3) $\lambda = \frac{1}{10}$ 1/min

Type 1: $p_1 = \frac{1}{3}$, $x_1 \sim Exp(\frac{1}{2})$, $\bar{x}_1 = 2$ min, $\mu_1 = \frac{1}{2}$
Type 2: $p_2 = \frac{2}{3}$, $x_2 \sim Exp(\frac{1}{5})$, $\bar{x}_2 = 5$ min, $\mu_2 = \frac{1}{5}$
$\Big\}\ \bar{x} = 4$

a) $M/H_2/1$ $(M/G/1)$

Secretary busy $= \rho = \lambda \bar{x} = 0.4$

b) $$W = \frac{\lambda \overline{x^2}}{2(1-\rho)} = \frac{\frac{1}{10}\cdot 36}{2\cdot\frac{6}{10}} = 3$$

$$\overline{x^2} = p_1 \overline{x_1^2} + p_2 \overline{x_2^2} = \frac{1}{3}\cdot\frac{2}{(1/2)^2} + \frac{2}{3}\,\frac{2}{(1/5)^2} = \frac{108}{3} = 36$$

$N_q = \lambda W = 0.3$

c) T_1: system time, when, at arrival, there is one customer in the system

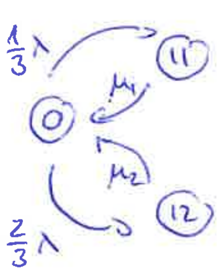
$T_1 = \bar{R}_s + \bar{x}$, $\bar{x} = 4$

$$\bar{R}_s = \{\text{see } M/G/1 \text{ lecture}\} = \frac{\sum_i \frac{1}{2} x_i^2}{\sum_i x_i} = \frac{\frac{1}{n}\sum \frac{1}{2} x_i^2}{\frac{1}{n}\sum x_i} = \frac{1}{2}\frac{\overline{x^2}}{\bar{x}} = 4.5$$

$T_1 = 4 + 4.5 = 8.5$ min

P(student needs to wait more than 5 minutes) =
$p_1 \cdot (x_1 > 5) + p_2 (x_2 > 5) = \frac{1}{3} e^{-\frac{1}{2} 5} + \frac{2}{3} e^{-\frac{1}{5}\cdot 5}$

d) $M/H_2/1/1$



$$\left.\begin{aligned} p_0 \cdot \tfrac{1}{3}\lambda &= p_{11} \cdot \mu_1 \\ p_0 \cdot \tfrac{2}{3}\lambda &= p_{12}\,\mu_2 \\ p_0 + p_{11} + p_{12} &= 1 \end{aligned}\right\} \quad [p_0, p_{11}, p_{12}] = [\tfrac{30}{42}, \tfrac{2}{42}, \tfrac{10}{42}]$$

P(leave without service) $= 1 - p_0 = \frac{12}{42} = \frac{2}{7}$

4) Preemptive resume priority, $M/G/1$

$\lambda = 2/\text{hour}$

Prio 1: $p_1 = \frac{1}{4} \Rightarrow \lambda_1 = \frac{1}{2}/\text{hour}$, $x_1 \sim Exp(4)$, $\bar{x} = \frac{1}{4}$, $\rho_1 = \frac{1}{8}$

Prio 2: $p_2 = \frac{3}{4} \Rightarrow \lambda_2 = \frac{3}{2}/\text{hour}$, $x_2 = \frac{1}{2}$, $\rho_2 = \frac{3}{4}$

a) $T_1 = \dfrac{(1-\rho_1)\bar{x}_1 + R_1}{(1-\rho_1)} = \dots = \frac{8}{28} = \frac{2}{7}$ $R_1 = \frac{1}{2}\lambda_1 \overline{x_1^2} = \frac{1}{32}$ hours

(Note, you should get the same result with $M/M/1$:

$T_1 = \dfrac{\rho}{\mu - \lambda} + \bar{x} = \dots = \frac{2}{7}$)

$T_2 = \dfrac{(1-\rho_1-\rho_2)\bar{x}_2 + R_2}{(1-\rho_1)(1-\rho_1-\rho_2)} =$

$= \dots = \frac{18}{7}$ hours

$R_2 = \frac{1}{2}(\lambda_1 \overline{x_1^2} + \lambda_2 \overline{x_2^2}) =$

$\frac{1}{2}(\frac{1}{16} + \frac{3}{2}\frac{1}{4}) = \frac{7}{32}$

For arbitrary application: $\bar{T} = p_1 T_1 + p_2 T_2 = \dots = 2$ hours

b) Same as for $M/M/1$

$N_q = \dfrac{\rho}{1-\rho} - \rho = \dots = \frac{1}{56}$

P(more than 2 high priority waiting) =

= P(more than 2 waiting in an $M/M/1$) =

$= 1 - (p_0 + p_1 + p_2 + p_3) = \dots = \frac{1}{4096} = 2.44 \cdot 10^{-4}$ $p_i = (1-\rho)\rho^i$

(4) Cont.

c) ~~Exam~~

P_2 arrives, P_1 under service,

P(start after the processing of the current) =

P(no new P_1 during the processing of the P_1) = $\frac{\mu_1}{\lambda_1+\mu_1} = \frac{4}{\frac{1}{2}+4} = \frac{8}{9}$

P(processing of P_2 interrupted) =

$1 -$ P(no P_1 arrives during the processing of P_2) =

$1 -$ P(no P_1 arrives within $\frac{1}{2}$ hour) $= 1 - e^{-\lambda_1 x_2} = 1 - e^{-\frac{1}{2}\cdot\frac{1}{2}} =$

$1 - e^{-\frac{1}{4}} = 0.22$

d) M/G/1 approximation:

P_1 overestimated

P_2 underestimated

Average: in the actual system shorter processing time requests do not wait, this decreases the average system time as well. Therefore M/G/1 approx. overestimates the system time of the priority system.

[Calculation:

$$T = \frac{\lambda(\overline{x^2})}{2(1-\rho)} + \bar{x} = \frac{\cancel{2}\cdot\frac{7}{32}}{\cancel{2}\cdot\frac{1}{8}} + \frac{7}{16} =$$

$$= \frac{7}{4} + \frac{7}{16} = \frac{28+7}{16} = \frac{35}{16} > \underline{\underline{2}}$$

$\rho = \rho_1 + \rho_2 = \frac{7}{8}$ $\qquad \lambda x = \rho$

$\bar{x} = \frac{\rho}{\lambda} = \frac{7}{16}$ $\qquad \left(\frac{1}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{2} = \frac{1}{16} + \frac{6}{16}\right.$

$\overline{x^2} = \frac{1}{4}\cdot\frac{\cancel{2}}{\cancel{16}_8} + \frac{3}{4}\cdot\frac{1}{4} = \frac{1}{32} + \frac{3}{16} = \frac{7}{32}$

5)

a) M/M/1 Departure process

i) continuous service: inter dep. time = service time $\Rightarrow$

$$\mathcal{L}(f(\tau_1)) = \frac{\mu}{s+\mu}$$

ii) empty queue: inter dep time = inter arrival time + service time $\Rightarrow$

$$\mathcal{L}(f(\tau_2)) = \frac{\lambda}{s+\lambda} \cdot \frac{\mu}{s+\mu}$$

P(empty queue) $= 1-\rho = 1-\frac{\lambda}{\mu}$, P(cont. service) = P(~~f~~ busy que) $= \frac{\lambda}{\mu}$

$$\mathcal{L}(f(\tau)) = P(\text{busy}) \cdot \mathcal{L}(f(\tau_1)) + P(\text{empty})\mathcal{L}(f(\tau_2)) =$$

$$\frac{\lambda}{\mu}\frac{\mu}{s+\mu} + \left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{s+\mu}\frac{\mu}{s+\mu} = \dots = \frac{\lambda}{s+\lambda}$$

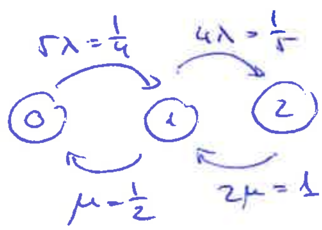
$\Rightarrow$ Inter departure time: $\tau \sim Exp(\lambda)$, $\bar{\tau} = \frac{1}{\lambda}$

Departure process: Poisson (λ)

b) M/M/2/2/5

$x \sim Exp(\mu)$ $\mu = \frac{1}{2}$ $\bar{x} = 2$

$\tau = Exp(\lambda)$ $\lambda = \frac{1}{20}$ $\bar{\tau} = 20$



$$\left.\begin{array}{l} p_0 5\lambda = p_1 \cdot \mu \\ p_1 \cdot 4\lambda = p_2 \cdot 2\mu \\ p_0 + p_1 + p_2 = 1 \end{array}\right\} \quad [p_0, p_1, p_2] = \left[\frac{10}{16}, \frac{5}{16}, \frac{1}{16}\right]$$

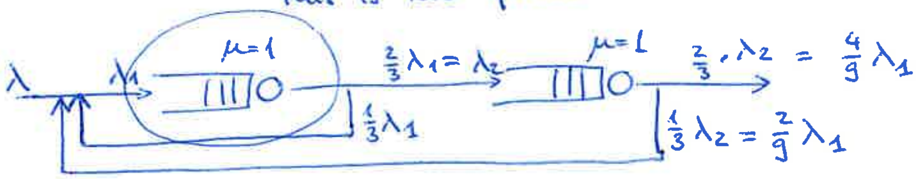
P(both printers busy) = {time blocking} $= p_2 = \frac{1}{16} = 0.0625$

P(employee turns back) = {call blocking} =

$$\frac{3\lambda p_2}{5\lambda p_0 + 4\lambda p_1 + 3\lambda p_2} = \dots = \frac{3}{73} = 0.041$$

5) cont'd

this is the queue with the highest load.

c) 

i) Stability: $\lambda_1 < \mu \Rightarrow \lambda_1 < 1$

$$\lambda_1 = \lambda + \underbrace{\left(\frac{1}{3}+\frac{2}{9}\right)}_{5/9}\lambda_1$$

$$\lambda_1\left(1-\frac{5}{9}\right) = \lambda$$

$$\lambda_1 = \frac{9}{4}\lambda < 1 \Rightarrow \boxed{\lambda < \frac{4}{9}}$$

ii) $\lambda = \frac{1}{4}$ $\qquad T = \dfrac{N_1+N_2}{\lambda}$

$\cancel{\rho_1 = \frac{\lambda_1}{\mu}} =$ $\lambda_1 = \frac{1}{4}\cdot\frac{9}{4} = \frac{9}{16}$ $\qquad \rho_1 = \frac{9}{16}$ $\qquad N_1 = \dfrac{9/16}{1-9/16} = \frac{9}{7}$

$\lambda_2 = \frac{2}{3}\lambda_1 = \frac{3}{8}$ $\qquad \rho_2 = \frac{3}{8}$ $\qquad N_2 = \dfrac{3/8}{1-3/8} = \frac{3}{5}$

$$T = \frac{\frac{9}{7}+\frac{3}{5}}{\frac{1}{4}} = \frac{45+21}{35}\cdot 4 = \frac{66\cdot 4}{35} = \frac{264}{35} \approx 7.06$$

d) M/G/1 with vacation

$V_1 = 1.5\,ms$ $\qquad \overline{V_1^2} = V_1^2$ $\qquad W_{M/G/1+V} = W_{M/G/1} + \dfrac{\overline{V^2}}{2\overline{V}}$

$V_2 \sim Exp(\mu)$ $\qquad \overline{V_2} = 1\,ms$ $\qquad \overline{V_2^2} = 2\cdot\overline{V_2}^2$

$$\boxed{\text{Type 1: } \frac{\overline{V^2}}{2\overline{V}} = \frac{1.5}{2} = 0.75\,ms}$$

⇐ This solution leads to lower average waiting time!

Type 2: $\qquad = \dfrac{2\cdot\overline{V_2}^2}{2\,\overline{V_2}} = \overline{V_2} = 1\,ms$