Interpretation of lead-lag controllers and their connection to PID controllers

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The aim here is to give a simple understanding of how the lead-lag controller works and how the lead and the lag part interact. The explanation is based on the PD-, and PI-controller and how they can be approximated. The lead-lag controller discussed is shown in Figure 1 showing how the controller may be represented by a low-pass filter feeding into a PD-controller which in turn feeds into a PI-controller.

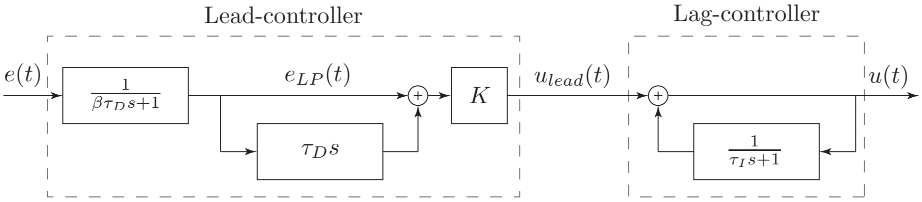

Figure 1: A lead-lag controller

1 The lead-controller

The lead-controller is a low-pass filter followed by a PD-controller.

1.1 The low-pass filter

We note that when $\beta\tau_D s$, were $0 \leq \beta \leq 1$, is small the following hold

$$\frac{1}{1+\beta\tau_D s} = \sum_{n=0}^{\infty}(-\beta\tau_D s)^n \approx 1 - \beta\tau_D s.$$

From this we may approximate the low-pass filter to be

$$e_{LP} = \frac{1}{\beta\tau_D s + 1} e \approx (1 - \beta\tau_D s)u \Rightarrow$$

$$e_{LP}(t) \approx (1-\beta)e(t) + \beta e(t) - \beta\tau_D \frac{d}{dt} e(t)$$

We note that this is the first term in the Taylor expansion so that for small τ_D we have

$$e_{LP}(t) \approx (1-\beta)e(t) + \beta e(t-\tau_D), \quad 0 \leq \beta \leq 1$$

1.2 The PD-controller

The PD-controller part is given by

$$u_{lead} = K\Big(e_{LP} + \tau_D s e_{LP}\Big) \quad \Rightarrow \quad u_{lead}(t) = K\Big(e_{LP}(t) + \tau_D \frac{d}{dt} e_{LP}(t)\Big)$$

We note that this is the first term in the Taylor expansion so that for small T_D we have

$$u_{lead}(t) \approx K e_{LP}(t + \tau_D)$$

1.3 The lead-controller

In summary this gives

$$u_{lead}(t) \approx K\Big((1-\beta)e(t+\tau_D) + \beta e(t)\Big), \quad 0 \le \beta \le 1$$

2 The lag-controller

The lag-controller is a regular PI-controlller giving u in Figure 1

$$u = u_{lead} + \frac{1}{\tau_I s + 1} u \quad \Rightarrow \quad u \quad = u_{lead} + \frac{1}{\tau_I s} u_{lead}$$

We note that when $\tau_I s$ is small the following hold

$$\frac{1}{1+\tau_I s} = \sum_{n=0}^{\infty} (-\tau_I s)^n \approx 1 - \tau_I s.$$

From this we may approximate the controller to be

$$u = u_{lead} + \frac{1}{\tau_I s + 1} u \approx u_{lead} + (1 - \tau_I s) u \Rightarrow$$
$$u(t) \approx u_{lead}(t) + u(t) - \tau_I \frac{d}{dt} u(t)$$

We note that this is the first term in the Taylor expansion so that for small τ_I we have

$$u(t) \approx u_{lead}(t) + u(t - \tau_I).$$

3 The lead-lag controller

In summary we have that

$$u(t) \approx u_{lead}(t) + u(t - \tau_I).$$
$$u_{lead}(t) \approx K e_{LP}(t + \tau_D)$$
$$e_{LP}(t) \approx (1-\beta)e(t) + \beta e(t - \tau_D), \quad 0 \le \beta \le 1$$

giving

$$u(t) \approx K\Big((1-\beta)e(t+\tau_D) + \beta e(t)\Big) + u(t - \tau_I), \quad 0 \le \beta \le 1.$$

4 Relations to PID-control

Here conversions between lead-lag and PID are shown.

4.1 lead-lag to PID

For a lead-lag controller given by

$$C_{ll}(s) = K \frac{\tau_I s + 1}{\tau_I s} \frac{\tau_D s + 1}{\beta \tau_D s + 1}$$

the corresponding PID-controller is given by

$$C_{PID}(s) = K_{PID}\Big(1 + \frac{1}{T_I s} + sT_D\Big)\frac{1}{T_F s + 1}$$

where

$$\begin{aligned} K_{PID} &= K\frac{\tau_I + \tau_D}{\tau_I} \\ T_I &= \tau_I + \tau_D \\ T_D &= \frac{\tau_I \tau_D}{\tau_I + \tau_D} \\ T_F &= \beta \tau_I \end{aligned}$$

4.2 PID to lead-lag

For a PID-controller given by

$$C_{PID}(s) = K_{PID}\Big(1 + \frac{1}{T_I s} + sT_D\Big)\frac{1}{T_F s + 1}$$

the corresponding lead-lag controller given by

$$C_{ll}(s) = K \frac{\tau_I s + 1}{\tau_I s} \frac{\tau_D s + 1}{\beta \tau_D s + 1}$$

where

$$\begin{aligned} K &= \frac{K_{PID}}{2}\Big(1 + \sqrt{1 - 4T_D/T_I}\Big) \\ \tau_I &= \frac{T_I}{2}\Big(1 + \sqrt{1 - 4T_D/T_I}\Big) \\ \tau_D &= \frac{T_I}{2}\Big(1 - \sqrt{1 - 4T_D/T_I}\Big) \\ \beta &= \frac{T_F}{\tau_I} \end{aligned}$$

Note that this conversion is only possible if

$$4T_D \leq T_I$$

and only has an interpretation as a lead-lag controller if

$$0 \leq \beta \leq 1$$