Probability Theory

Ω is a set called the sample space

ω a point of Ω is called a sample point

$\mathcal{F}$ is a collection of subsets of Ω. A member of $\mathcal{F}$ is called an event

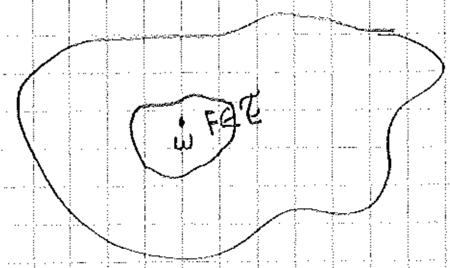
$P : \mathcal{F} \to [0,1]$ is a set function satisfying

i) $F_\alpha \in \mathcal{F}$ $\alpha \in A$ The F_α's disjoint $P(\bigcup_{\alpha \in A} F_\alpha) = \bigcup_{\alpha \in A} P(F_\alpha)$

ii) $P(\Omega) = 1$

P is called the probability measure (p.m.), if $P(\Omega) \neq 1$, P is called a measure

$(\Omega, \mathcal{F}, P)$ is called a probability space (triple)



Ω

Structure of $\mathcal{F}$

We would like to have $\mathcal{F} = \{\text{all subsets of } \Omega\}$

If the number of ~~on~~ sample points is uncountable, this is an incredibly rich collection of sets, and this leads to problems when defining a p.m.

Ex 1 Ω = Unit sphere in R^3 $(= S^2)$ $P(F) = \frac{\text{Area}(F)}{\text{Area}(S^2)}$

Banach & Tarski : $\exists F$ & Rotations $F_{i,k}$ of F. s.t.

$$P(F) = P(F_{i,k})$$

$$F_{i,k} \cap F_{j,k} = \emptyset \quad i \neq j$$

$$S^2 = \bigcup_{i=1}^{k} F_{i,k} \quad k \geq 3$$

$$\Rightarrow \quad 1 = k \cdot P(F) \quad k \geq 3$$

$\therefore$ F has different areas.

Natural Requirements

I) $\Omega \in \mathcal{F}$

II) $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$

III) $F, G \in \mathcal{F} \Rightarrow F \cup G \in \mathcal{F}$

An $\mathcal{F}$ satisfying i) - iii) is called an algebra

II) + III) $\Rightarrow F, G \in \mathcal{F} \Rightarrow F \cap G \in \mathcal{F}$

III) $\Rightarrow \{F_k\}_{k=1}^{N} \in \mathcal{F} \Rightarrow \bigcup_{k=1}^{N} F_k \in \mathcal{F}$ if $N < \infty$

Limits

$\hat{\Theta}_N$ estimator of Θ based on $X_N = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$

$$F(\varepsilon) = \{ \omega : \limsup_{N \to \infty} |\hat{\Theta}_N - \Theta| \leq \varepsilon \}$$

$$= \{ \omega ; \ |\hat{\Theta}_N - \Theta| \leq \varepsilon \text{ for } N \geq N(\omega) \}$$

$$= \{ \omega : \ |\hat{\Theta}_N - \Theta| \leq \varepsilon \text{ for } N \text{ suff. large} \}$$

$$= \{ \omega : \ |\hat{\Theta}_N - \Theta| \leq \varepsilon \text{ eventually} \}$$

What is necessary for $F(\varepsilon)$ to be an event?

Let $F_N(\varepsilon) = \{ \omega ; \ |\hat{\Theta}_N - \Theta| \leq \varepsilon \}$

$$\Rightarrow F(\varepsilon) = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} F_n(\varepsilon) \quad \left(= \liminf F_n(\varepsilon) = F_n(\varepsilon) \text{ ev} \right)$$

$\Rightarrow$ IV) $\{F_k\}_1^{\infty} \in \mathcal{F} \Rightarrow \bigcup_{k=1}^{\infty} F_k \in \mathcal{F}$ a necessary condition on $\mathcal{F}$

An algebra $\mathcal{F}$ satisfying IV) is called a σ-algebra

The pair $(\Omega, \mathcal{F})$ is called a measurable space

Consequence

i') can be replaced by

i) $\{F_k\}_{k=1}^{N} \in \mathcal{F} \quad F_k \cap F_\ell = \emptyset \quad k \neq \ell \Rightarrow P(\bigcup_{k=1}^{N} F_k) = \sum_{k=1}^{N} P(F_k) \quad N \leq \infty$

A set function $\mu : \mathcal{F} \to [0, \infty]$ satisfying i) is called a measure. P is countably additive

Sub σ-algebras: let $\mathcal{F}$ & $\mathcal{G}$ σ-alg on Ω. If $G \in \mathcal{G} \Rightarrow G \in \mathcal{F}$ then $\mathcal{G}$ is a sub σ-alg of $\mathcal{F}$.

Generating σ-algebras

$\mathcal{C}$ a class of subsets on S

$\sigma(\mathcal{C})$ = smallest σ-algebra Σ on S s.t. $\mathcal{C} \subseteq \Sigma$

Borel σ-algebras

S topological space

$\mathcal{B}(S) = \sigma(\text{open sets})$ — sets in $\mathcal{B}(S)$ Borel sets

Ex2) $\mathcal{B} = \mathcal{B}(\mathbb{R})$

$$\pi(\mathbb{R}) = \{(-\infty, x] \ \ x \in \mathbb{R}\} \Rightarrow \mathcal{B}(\mathbb{R}) = \sigma(\pi(\mathbb{R}))$$

$$\Rightarrow (x,y) \ \ (x,y] \ \ [x,y] \in \mathcal{B}(\mathbb{R})$$

Carathéodory's Extension Theorem

S a set, Σ_0 an algebra on S, $\underbrace{\mu_0}_{\text{(finite)}}$ measure on Σ_0

$$\Sigma = \sigma(\Sigma_0)$$

Then $\exists$ (unique) measure μ on Σ s.t. $\mu = \mu_0$ on Σ_0

Lesbegue measure on $([0,1], \mathcal{B}([0,1]))$

$$F \in \Sigma_0 \text{ if } F = \bigcup_{k=1}^{N} (a_k, b_k] \qquad N \leq \infty$$

$$\Rightarrow \Sigma_0 \text{ algebra}, \quad \sigma(\Sigma_0) = \mathcal{B}([0,1])$$

$$\text{Define } \mu_0(F) = \sum_{k=1}^{N} (b_k - a_k) \qquad \text{satisfies 1)}$$

Theor $\Rightarrow \exists\ \mu$ on $([0,1], \mathcal{B}([0,1]))$ s.t.

$\mu(F) = \mu_0(F)$ if $F \in \Sigma_0$

μ is called the Lesbegue measure

μ is the general measure of length

Similar construction can be used e.g. on S^2. F in Ex1) does not belong to the Borel σ-algebra on S^2.

Monotone Convergence

μ measure

$F_n \in \Sigma \quad n = 1, 2, \dots \qquad F_n \uparrow F \quad \Rightarrow \quad \mu(F_n) \uparrow \mu(F)$

Proof: $\quad G_1 = F_1 \quad G_n = F_n \setminus F_{n-1} \quad n \geq 2$

$$\mu(F_n) = \mu\left(\bigcup_{k=1}^{n} G_k\right) = \sum_{k=1}^{n} \mu(G_k) \to \sum_{k=1}^{\infty} \mu(G_k) = \mu\left(\sum_{k=1}^{\infty} G_k\right) = \mu(F)$$

$F_n \downarrow F$, $\mu(F_k) < \infty$ for som $k \Rightarrow \mu(F_n) \downarrow \mu(F)$