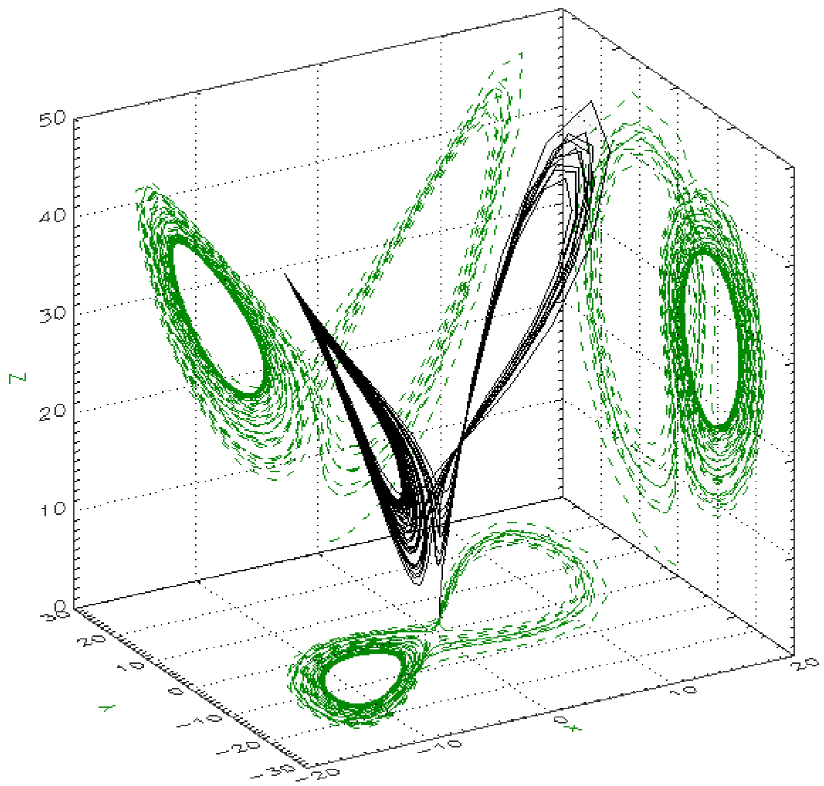
EL2620 Nonlinear Control

Lecture notes

Karl Henrik Johansson, Bo Wahlberg and Elling W. Jacobsen

This revision December 2012



Automatic Control
KTH, Stockholm, Sweden

Preface

Many people have contributed to these lecture notes in nonlinear control. Originally it was developed by Bo Bernhardsson and Karl Henrik Johansson, and later revised by Bo Wahlberg and myself. Contributions and comments by Mikael Johansson, Ola Markusson, Ragnar Wallin, Henning Schmidt, Krister Jacobsson, Björn Johansson and Torbjörn Nordling are gratefully acknowledged.

Elling W. Jacobsen
Stockholm, December 2012

EL2620 Nonlinear Control

Automatic Control Lab, KTH

- **Disposition**

  7.5 credits, *lp 2*

  28h lectures, 28h exercises, 3 home-works

- **Instructors**

  Elling W. Jacobsen, lectures and course responsible
  `jacobsen@kth.se`

  Farhad Farokhi, Martin Andreasson, teaching assistants
  `farakhi@kth.se`, `mandreas@kth.se`

  Hanna Holmqvist, course administration
  `hanna.holmqvist@ee.kth.se`

  STEX (entrance floor, Osquldasv. 10), course material
  `stex@s3.kth.se`

Course Goal

To provide participants with a solid theoretical foundation of nonlinear control systems combined with a good engineering understanding

You should after the course be able to

- understand common nonlinear control phenomena
- apply the most powerful nonlinear analysis methods
- use some practical nonlinear control design methods

EL2620 Nonlinear Control

Lecture 1

- Practical information
- Course outline
- Linear vs Nonlinear Systems
- Nonlinear differential equations

Today's Goal

You should be able to

- Define a nonlinear dynamic system
- Describe distinctive phenomena in nonlinear dynamic systems
- Transform differential equations to first-order form
- Derive equilibrium points

Course Information

- All info and handouts are available at course homepage (KTH Social)
- Homeworks are compulsory and have to be handed in on time (we are strict on this!)
- Written 5h exam on January 18 2014

Material

- **Textbook:** Khalil, *Nonlinear Systems*, Prentice Hall, 3rd ed., 2002. Optional but highly recommended.
- **Lecture notes:** Copies of transparencies (from previous year)
- **Exercises:** Class room and home exercises
- **Homeworks:** 3 computer exercises to hand in
- **Software:** Matlab / Simulink

Alternative textbooks (decreasing mathematical rigour):
Sastry, *Nonlinear Systems: Analysis, Stability and Control*; Vidyasagar, *Nonlinear Systems Analysis*; Slotine & Li, *Applied Nonlinear Control*; Glad & Ljung, *Reglerteori, flervariabla och olinjära metoder*.

Only references to Khalil will be given.

Two course compendia sold by STEX (also available at course homepage)

Course Outline

- **Introduction:** nonlinear models and phenomena, computer simulation (L1-L2)
- **Feedback analysis:** linearization, stability theory, describing functions (L3-L6)
- **Control design:** compensation, high-gain design, Lyapunov methods (L7-L10)
- **Alternatives:** gain scheduling, optimal control, neural networks, fuzzy control (L11-L13)
- **Summary** (L14)

Linear Systems

Definition: Let M be a signal space. The system $S : M \to M$ is linear if for all $u, v \in M$ and $\alpha \in \mathbb{R}$

$$S(\alpha u) = \alpha S(u) \qquad \text{scaling}$$
$$S(u+v) = S(u) + S(v) \qquad \text{superposition}$$

Example: Linear time-invariant systems

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = 0$$
$$y(t) = g(t) \star u(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$
$$Y(s) = G(s)U(s)$$

Notice the importance to have zero initial conditions

Linear Systems Have Nice Properties

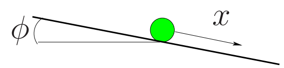
Local stability=global stability Stability if all eigenvalues of A (or poles of $G(s)$) are in the left half-plane

Superposition Enough to know a step (or impulse) response

Frequency analysis possible Sinusoidal inputs give sinusoidal outputs: $Y(i\omega) = G(i\omega)U(i\omega)$

Linear Models may be too Crude Approximations

Example: Positioning of a ball on a beam



Nonlinear model: $m\ddot{x}(t) = mg \sin \phi(t),$ Linear model: $\ddot{x}(t) = g\phi(t)$

Can the ball move 0.1 meter in 0.1 seconds from steady state?

Linear model (step response with $\phi = \phi_0$) gives

$$x(t) \approx 10\frac{t^2}{2}\phi_0 \approx 0.05\phi_0$$

so that

$$\phi_0 \approx \frac{0.1}{0.05} = 2 \text{ rad} = 114^\circ$$

Unrealistic answer. Clearly outside linear region!

Linear model valid only if $\sin \phi \approx \phi$

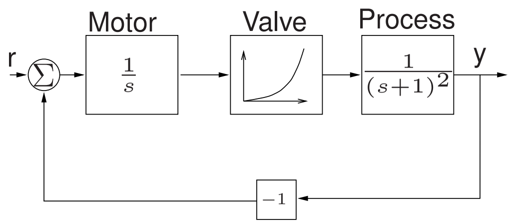
Must consider nonlinear model. Possibly also include other nonlinearities such as centripetal force, saturation, friction etc.

Linear Models are not Rich Enough

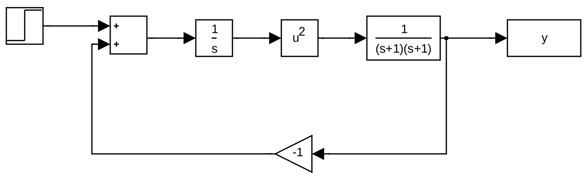
Linear models can not describe many phenomena seen in nonlinear systems

Stability Can Depend on Reference Signal

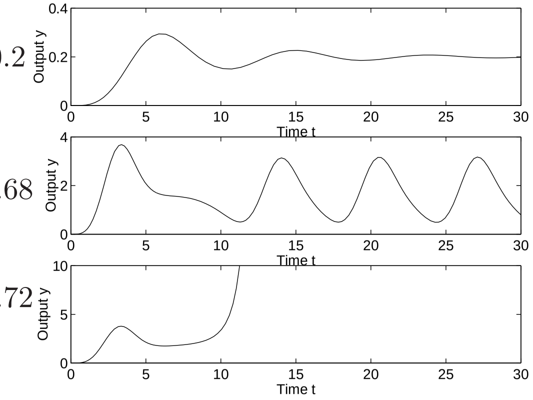
Example: Control system with valve characteristic $f(u) = u^2$



Simulink block diagram:



Step Responses



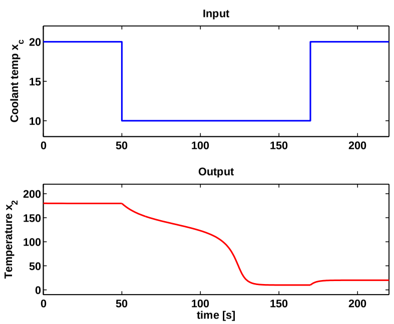
Stability depends on amplitude of the reference signal!

(The linearized gain of the valve increases with increasing amplitude)

Multiple Equilibria

Example: chemical reactor

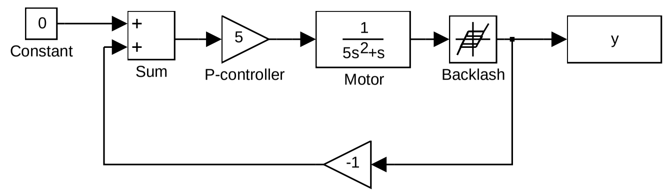
$$\begin{aligned} \dot{x}_1 &= -x_1 exp\left(-\frac{1}{x_2}\right) + f(1 - x_1) \\ \dot{x}_2 &= x_1 exp\left(-\frac{1}{x_2}\right) - \epsilon f(x_2 - x_c) \\ f &= 0.7, \epsilon = 0.4 \end{aligned}$$



Existence of multiple stable equilibria for the same input gives hysteresis effect

Stable Periodic Solutions

Example: Position control of motor with back-lash

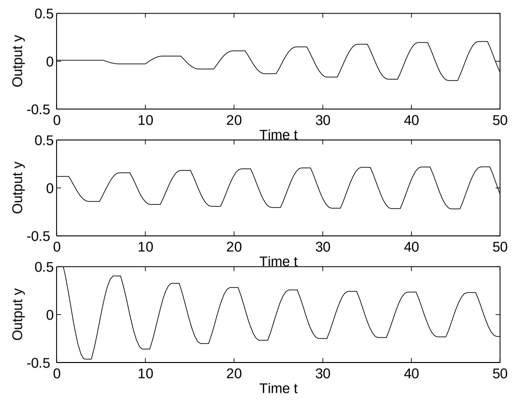


Motor: $G(s) = \frac{1}{s(1+5s)}$

Controller: $K = 5$

Back-lash induces an oscillation

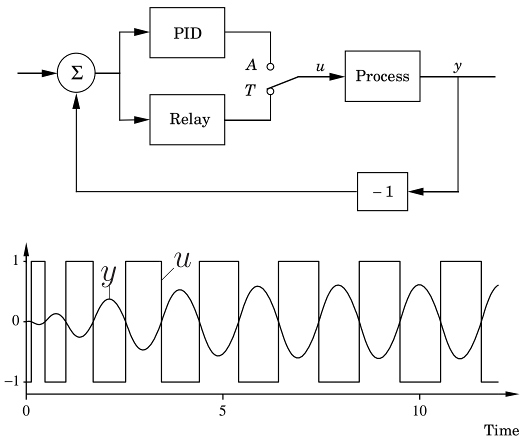
Period and amplitude independent of initial conditions:



How predict and avoid oscillations?

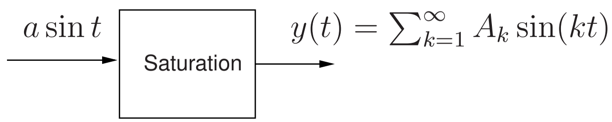
Automatic Tuning of PID Controllers

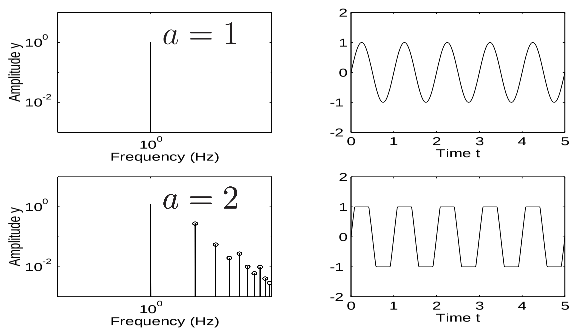
Relay induces a desired oscillation whose frequency and amplitude are used to choose PID parameters



Harmonic Distortion

Example: Sinusoidal response of saturation





Example: Electrical power distribution

Nonlinearities such as rectifiers, switched electronics, and transformers give rise to harmonic distortion

$$\text{Total Harmonic Distortion} = \frac{\sum_{k=2}^{\infty} \text{Energy in tone } k}{\text{Energy in tone 1}}$$

Example: Electrical amplifiers

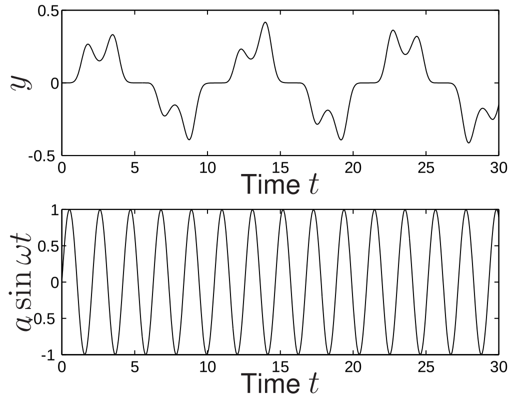
Effective amplifiers work in nonlinear region

Introduces spectrum leakage, which is a problem in cellular systems

Trade-off between effectivity and linearity

Subharmonics

Example: Duffing's equation $\ddot{y} + \dot{y} + y - y^3 = a\sin(\omega t)$



Nonlinear Differential Equations

Definition: A solution to

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \qquad (1)$$

over an interval $[0, T]$ is a $\mathbf{C}^1$ function $x : [0, T] \to \mathbb{R}^n$ such that (1) is fulfilled.

- When does there exists a solution?
- When is the solution unique?

Example: $\dot{x} = Ax$, $x(0) = x_0$, gives $x(t) = \exp(At)x_0$

Existence Problems

Example: The differential equation $\dot{x} = x^2$, $x(0) = x_0$

has solution $\quad x(t) = \dfrac{x_0}{1 - x_0 t}, \qquad 0 \le t < \dfrac{1}{x_0}$

Solution not defined for $\quad t_f = \dfrac{1}{x_0}$

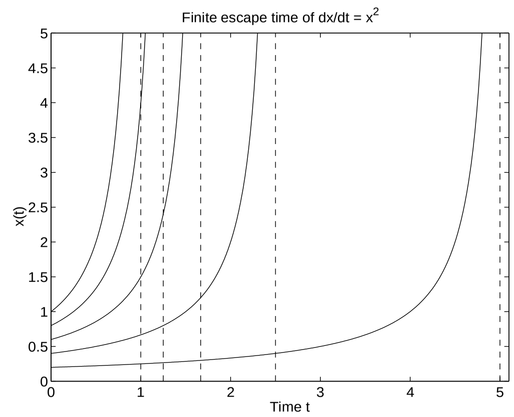
Solution interval depends on initial condition!

Recall the trick: $\dot{x} = x^2 \;\Rightarrow\; \dfrac{dx}{x^2} = dt$

Integrate $\Rightarrow \dfrac{-1}{x(t)} - \dfrac{-1}{x(0)} = t \;\Rightarrow\; x(t) \;=\; \dfrac{x_0}{1 - x_0 t}$

Finite Escape Time

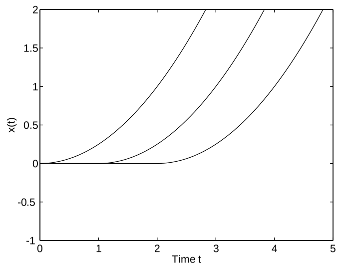
Simulation for various initial conditions x_0



Uniqueness Problems

Example: $\dot{x} = \sqrt{x}$, $x(0) = 0$, has many solutions:

$$x(t) = \begin{cases} (t-C)^2/4 & t > C \\ 0 & t \le C \end{cases}$$



Physical Interpretation

Consider the reverse example, i.e., the water tank lab process with

$$\dot{x} = -\sqrt{x}, \qquad x(T) = 0$$

where x is the water level. It is then impossible to know at what time $t < T$ the level was $x(t) = x_0 > 0$.

Hint: Reverse time $s = T - t \Rightarrow ds = -dt$ and thus

$$\frac{dx}{ds} = -\frac{dx}{dt}$$

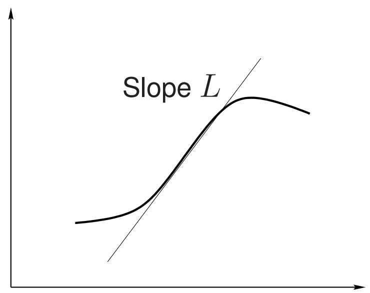
Lipschitz Continuity

Definition: $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous if there exists $L, r > 0$ such that for all $x, y \in B_r(x_0) = \{z \in \mathbb{R}^n : \|z - x_0\| < r\}$,

$$\|f(x) - f(y)\| \le L\|x - y\|$$

Euclidean norm is given by

$$\|x\|^2 = x_1^2 + \cdots + x_n^2$$



Local Existence and Uniqueness

Theorem:
If f is Lipschitz continuous, then there exists $\delta > 0$ such that

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0$$

has a unique solution in $B_r(x_0)$ over $[0, \delta]$. **Proof:** See Khalil, Appendix C.1. Based on the contraction mapping theorem

Remarks

- $\delta = \delta(r, L)$
- f being C^0 is not sufficient (cf., tank example)
- f being C^1 implies Lipschitz continuity ($L = \max_{x \in B_r(x_0)} f'(x)$)

State-Space Models

State x, input u, output y

General: $f(x, u, y, \dot{x}, \dot{u}, \dot{y}, \dots) = 0$

Explicit: $\dot{x} = f(x, u), \quad y = h(x)$

Affine in u: $\dot{x} = f(x) + g(x)u, \quad y = h(x)$

Linear: $\dot{x} = Ax + Bu, \quad y = Cx$

Transformation to Autonomous System

A nonautonomous system

$$\dot{x} = f(x, t)$$

is always possible to transform to an autonomous system by introducing $x_{n+1} = t$:

$$\begin{aligned} \dot{x} &= f(x, x_{n+1}) \\ \dot{x}_{n+1} &= 1 \end{aligned}$$

Transformation to First-Order System

Given a differential equation in y with highest derivative $\frac{d^n y}{dt^n}$, express the equation in $x = \begin{pmatrix} y & \frac{dy}{dt} & \dots & \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}^T$

Example: Pendulum

$$MR^2\ddot{\theta} + k\dot{\theta} + MgR\sin\theta = 0$$

$x = \begin{pmatrix} \theta & \dot{\theta} \end{pmatrix}^T$ gives

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{MR^2}x_2 - \frac{g}{R}\sin x_1 \end{aligned}$$

Equilibria

Definition: A point (x^*, u^*, y^*) is an equilibrium, if a solution starting in (x^*, u^*, y^*) stays there forever.

Corresponds to putting all derivatives to zero:

General: $f(x^*, u^*, y^*, 0, 0, \dots) = 0$

Explicit: $0 = f(x^*, u^*), \quad y^* = h(x^*)$

Affine in u: $0 = f(x^*) + g(x^*)u^*, \quad y^* = h(x^*)$

Linear: $0 = Ax^* + Bu^*, \quad y^* = Cx^*$

Often the equilibrium is defined only through the state x^*

Multiple Equilibria

Example: Pendulum

$$MR^2\ddot{\theta} + k\dot{\theta} + MgR\sin\theta = 0$$

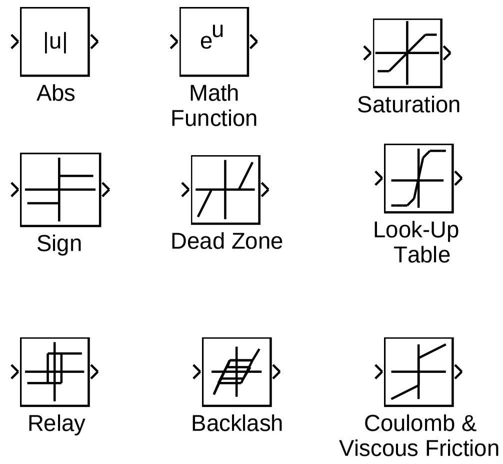
$\ddot{\theta} = \dot{\theta} = 0$ gives $\sin\theta = 0$ and thus $\theta^* = k\pi$

Alternatively in first-order form:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{MR^2}x_2 - \frac{g}{R}\sin x_1$$

$\dot{x}_1 = \dot{x}_2 = 0$ gives $x_2^* = 0$ and $\sin(x_1^*) = 0$

Some Common Nonlinearities in Control Systems



When do we need Nonlinear Analysis & Design?

- When the system is strongly nonlinear, or not linearizable
- When the range of operation is large
- When distinctive nonlinear phenomena are relevant
- When we want to push performance to the limit

Next Lecture

- Simulation in Matlab
- Linearization
- Phase plane analysis

EL2620 Nonlinear Control

Lecture 2

- Wrap-up of Lecture 1: Nonlinear systems and phenomena
- Modeling and simulation in Simulink
- Phase-plane analysis

Today's Goal

You should be able to

- Model and simulate in Simulink
- Linearize using Simulink
- Do phase-plane analysis using pplane (or other tool)

Analysis Through Simulation

Simulation tools:

ODE's $\dot{x} = f(t, x, u)$

- ACSL, Simnon, Simulink

DAE's $F(t, \dot{x}, x, u) = 0$

- Omsim, Dymola, Modelica
  `http://www.modelica.org`

Special purpose simulation tools

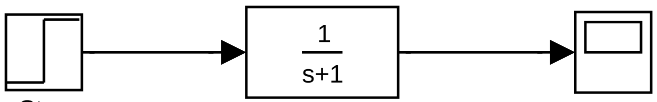
- Spice, EMTP, ADAMS, gPROMS

Simulink

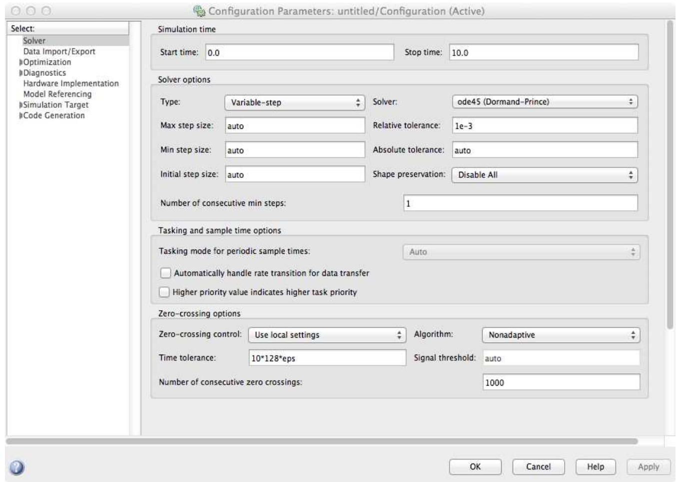
```
> matlab
>> simulink
```

An Example in Simulink

```
File -> New -> Model
Double click on Continous
Transfer Fcn
Step (in Sources)
Scope (in Sinks)
Connect (mouse-left)
Simulation -> Parameters
```



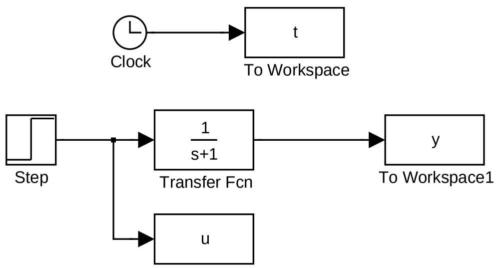
Choose Simulation Parameters



Don't forget "Apply"

Save Results to Workspace

stepmodel.mdl



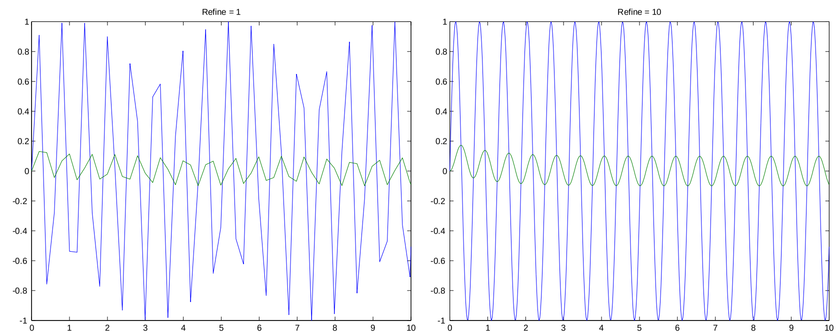
Check "Save format" of output blocks ("Array" instead of "Structure")

```
>> plot(t,y)
```


How To Get Better Accuracy

Modify Refine, Absolute and Relative Tolerances, Integration method

Refine adds interpolation points:



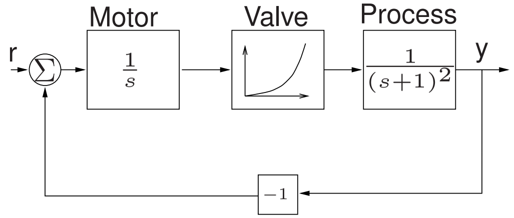
Use Scripts to Document Simulations

If the block-diagram is saved to `stepmodel.mdl`,
the following Script-file `simstepmodel.m` simulates the system:

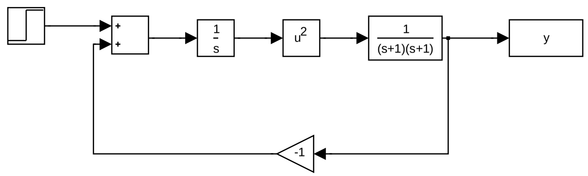
```
open_system('stepmodel')
set_param('stepmodel','RelTol','1e-3')
set_param('stepmodel','AbsTol','1e-6')
set_param('stepmodel','Refine','1')
tic
sim('stepmodel',6)
toc
subplot(2,1,1),plot(t,y),title('y')
subplot(2,1,2),plot(t,u),title('u')
```


Nonlinear Control System

Example: Control system with valve characteristic $f(u) = u^2$



Simulink block diagram:

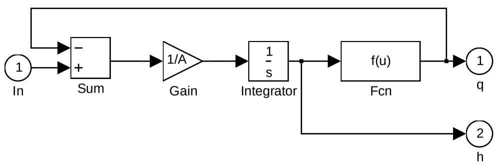


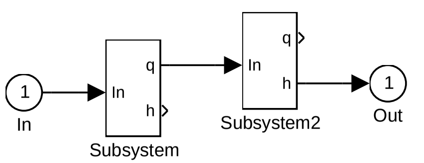
Example: Two-Tank System

The system consists of two identical tank models:

$$\dot{h} = (u - q)/A$$

$$q = a\sqrt{2g}\sqrt{h}$$





Linearization in Simulink

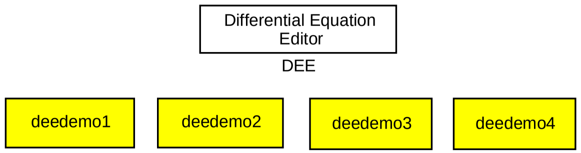
Linearize about equilibrium (x_0, u_0, y_0):

```
>> A=2.7e-3;a=7e-6,g=9.8;
>> [x0,u0,y0]=trim('twotank',[0.1 0.1]',[],0.1)
x0 =
   0.1000
   0.1000
u0 =
  9.7995e-006
y0 =
   0.1000
>> [aa,bb,cc,dd]=linmod('twotank',x0,u0);
>> sys=ss(aa,bb,cc,dd);
>> bode(sys)
```

Differential Equation Editor

`dee` is a Simulink-based differential equation editor

```
>> dee
```



Run the demonstrations

Phase-Plane Analysis

- Download ICTools from
  `http://www.control.lth.se/˜ictools`
- Down load DFIELD and PPLANE from
  `http://math.rice.edu/˜dfield`
  This was the preferred tool last year!

Homework 1

- Use your favorite phase-plane analysis tool
- Follow instructions in Exercise Compendium on how to write the report.
- See the course homepage for a report example
- The report should be short and include only necessary plots. Write in English.

EL2620 Nonlinear Control

Lecture 3

- Stability definitions
- Linearization
- Phase-plane analysis
- Periodic solutions

Today's Goal

You should be able to

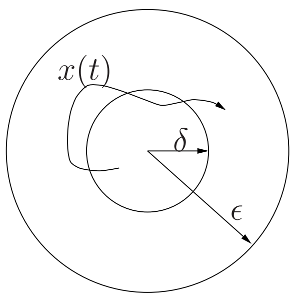
- Explain local and global stability
- Linearize around equilibria and trajectories
- Sketch phase portraits for two-dimensional systems
- Classify equilibria into nodes, focuses, saddle points, and center points
- Analyze stability of periodic solutions through Poincaré maps

Local Stability

Consider $\dot{x} = f(x)$ with $f(0) = 0$

Definition: The equilibrium $x^* = 0$ is **stable** if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq 0$$



If $x^* = 0$ is not stable it is called **unstable**.

Asymptotic Stability

Definition: The equilibrium $x = 0$ is **asymptotically stable** if it is stable and δ can be chosen such that

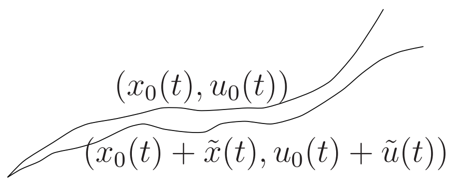
$$\|x(0)\| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0$$

The equilibrium is **globally asymptotically stable** if it is stable and $\lim_{t\to\infty} x(t) = 0$ for all $x(0)$.

Linearization Around a Trajectory

Let $(x_0(t), u_0(t))$ denote a solution to $\dot{x} = f(x,u)$ and consider another solution $(x(t), u(t)) = (x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$:

$$\begin{aligned}\dot{x}(t) &= f(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t)) \\ &= f(x_0(t), u_0(t)) + \frac{\partial f}{\partial x}(x_0(t), u_0(t))\tilde{x}(t) \\ &\quad + \frac{\partial f}{\partial u}(x_0(t), u_0(t))\tilde{u}(t) + \mathcal{O}(\|\tilde{x}, \tilde{u}\|^2)\end{aligned}$$



Hence, for small $(\tilde{x}, \tilde{u})$, approximately

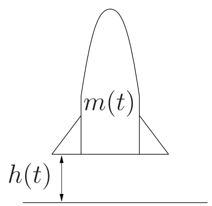
$$\dot{\tilde{x}}(t) = A(x_0(t), u_0(t))\tilde{x}(t) + B(x_0(t), u_0(t))\tilde{u}(t)$$

where

$$\begin{aligned}A(x_0(t), u_0(t)) &= \frac{\partial f}{\partial x}(x_0(t), u_0(t)) \\ B(x_0(t), u_0(t)) &= \frac{\partial f}{\partial u}(x_0(t), u_0(t))\end{aligned}$$

Note that A and B are time dependent. However, if $(x_0(t), u_0(t)) \equiv (x_0, u_0)$ then A and B are constant.

Example



$$\begin{aligned}\dot{h}(t) &= v(t) \\ \dot{v}(t) &= -g + v_e u(t)/m(t) \\ \dot{m}(t) &= -u(t)\end{aligned}$$

Let $x_0(t) = (h_0(t), v_0(t), m_0(t))^T$, $u_0(t) \equiv u_0 > 0$, be a solution. Then,

$$\dot{\tilde{x}}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\frac{v_e u_0}{(m_0 - u_0 t)^2} \\ 0 & 0 & 0 \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} 0 \\ \frac{v_e}{m_0 - u_0 t} \\ 1 \end{pmatrix} \tilde{u}(t)$$

Pointwise Left Half-Plane Eigenvalues of $A(t)$ Do *Not* Impose Stability

$$A(t) = \begin{pmatrix} -1 + \alpha\cos^2 t & 1 - \alpha \sin t \cos t \\ -1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t \end{pmatrix}, \quad \alpha > 0$$

Pointwise eigenvalues are given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

which are stable for $0 < \alpha < 2$. However,

$$x(t) = \begin{pmatrix} e^{(\alpha-1)t}\cos t & e^{-t}\sin t \\ -e^{(\alpha-1)t}\sin t & e^{-t}\cos t \end{pmatrix} x(0),$$

is unbounded solution for $\alpha > 1$.

Lyapunov's Linearization Method

Theorem: Let x_0 be an equilibrium of $\dot{x} = f(x)$ with $f \in \mathbb{C}^1$. Denote $A = \frac{\partial f}{\partial x}(x_0)$ and $\alpha(A) = \max \mathrm{Re}(\lambda(A))$.

- If $\alpha(A) < 0$, then x_0 is asymptotically stable
- If $\alpha(A) > 0$, then x_0 is unstable

The fundamental result for linear systems theory!

The case $\alpha(A) = 0$ needs further investigation.

The theorem is also called *Lyapunov's Indirect Method*.

A proof is given next lecture.

Example

The linearization of

$$\dot{x}_1 = -x_1^2 + x_1 + \sin x_2$$
$$\dot{x}_2 = \cos x_2 - x_1^3 - 5x_2$$

at the equilibrium $x_0 = (1, 0)^T$ is given by

$$A = \begin{pmatrix} -1 & 1 \\ -3 & -5 \end{pmatrix}, \qquad \lambda(A) = \{-2, -4\}$$

x_0 is thus an asymptotically stable equilibrium for the *nonlinear* system.

Linear Systems Revival

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analytic solution: $x(t) = e^{At}x(0), t \geq 0.$

If A is diagonalizable, then

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

where v_1, v_2 are the eigenvectors of A ($Av_1 = \lambda_1 v_1$ etc).

This implies that

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2,$$

where the constants c_1 and c_2 are given by the initial conditions

Example: Two real negative eigenvalues

Given the eigenvalues $\lambda_1 < \lambda_2 < 0$, with corresponding eigenvectors v_1 and v_2, respectively.

Solution: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

Slow eigenvalue/vector: $x(t) \approx c_2 e^{\lambda_2 t} v_2$ for large t.
Moves along the slow eigenvector towards $x = 0$ for large t

Fast eigenvalue/vector: $x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2$ for small t.
Moves along the fast eigenvector for small t

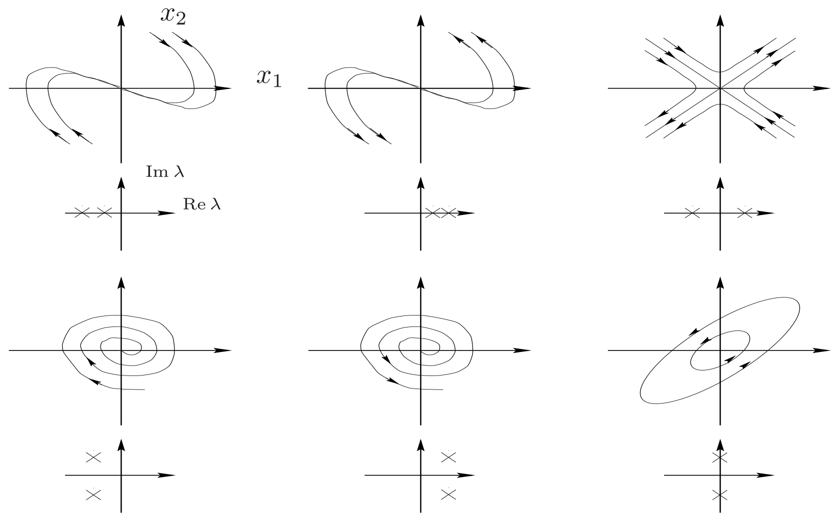
Phase-Plane Analysis for Linear Systems

The location of the eigenvalues $\lambda(A)$ determines the characteristics of the trajectories.

Six cases:

	stable node	unstable node	saddle point
$\operatorname{Im}\lambda_i = 0:$	$\lambda_1, \lambda_2 < 0$	$\lambda_1, \lambda_2 > 0$	$\lambda_1 < 0 < \lambda_2$
$\operatorname{Im}\lambda_i \neq 0:$	$\operatorname{Re}\lambda_i < 0$	$\operatorname{Re}\lambda_i > 0$	$\operatorname{Re}\lambda_i = 0$
	stable focus	unstable focus	center point

Equilibrium Points for Linear Systems



Example—Unstable Focus

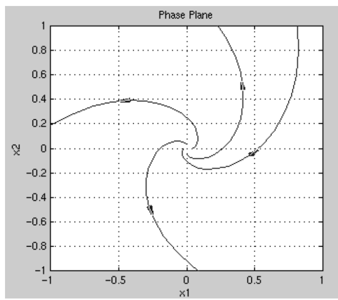
$$\dot{x} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} x, \qquad \sigma, \omega > 0, \qquad \lambda_{1,2} = \sigma \pm i\omega$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\sigma t}e^{i\omega t} & 0 \\ 0 & e^{\sigma t}e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} x(0)$$

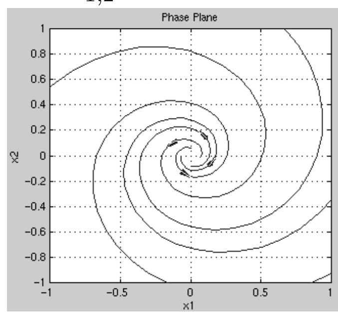
In polar coordinates $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan x_2/x_1$ ($x_1 = r\cos\theta$, $x_2 = r\sin\theta$):

$$\dot{r} = \sigma r$$
$$\dot{\theta} = \omega$$

$\lambda_{1,2} = 1 \pm i$



$\lambda_{1,2} = 0.3 \pm i$

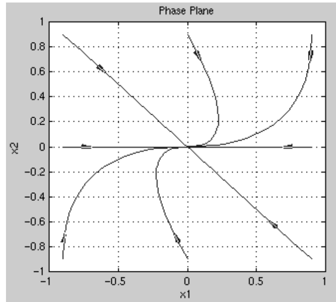


Example—Stable Node

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x$$

$$(\lambda_1, \lambda_2) = (-1, -2) \quad \text{and} \quad \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

v_1 is the slow direction and v_2 is the fast.



Fast: $x_2 = -x_1 + c_3$
Slow: $x_2 = 0$

5 minute exercise: *What is the phase portrait if* $\lambda_1 = \lambda_2$*?*

Hint: Two cases; only one linear independent eigenvector or all vectors are eigenvectors

Phase-Plane Analysis for Nonlinear Systems

Close to equilibrium points "nonlinear system" $\approx$ "linear system"

Theorem: Assume

$$\dot{x} = f(x) = Ax + g(x),$$

with $\lim_{\|x\|\to 0} \|g(x)\|/\|x\| = 0$. If $\dot{z} = Az$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin.

Remark: If the linearized system has a center, then the nonlinear system has either a center or a focus.

How to Draw Phase Portraits

By hand:

1. Find equilibria
2. Sketch local behavior around equilibria
3. Sketch $(\dot{x}_1, \dot{x}_2)$ for some other points. Notice that

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_1}{\dot{x}_2}$$

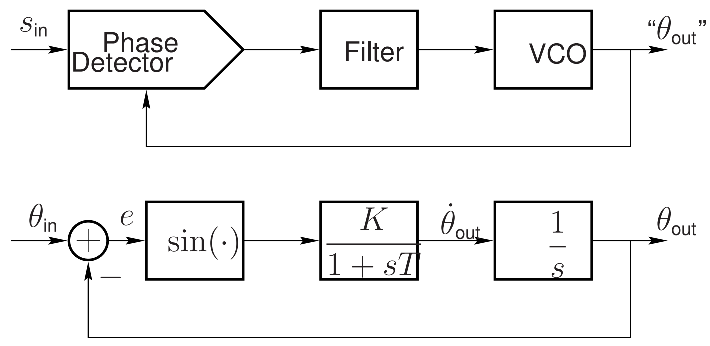
4. Try to find possible periodic orbits
5. Guess solutions

By computer:

1. Matlab: `dee` or `pplane`

Phase-Locked Loop

A PLL tracks phase $\theta_{\text{in}}(t)$ of a signal $s_{\text{in}}(t) = A\sin[\omega t + \theta_{\text{in}}(t)]$.



Phase-Plane Analysis of PLL

Let $(x_1, x_2) = (\theta_{\text{out}}, \dot{\theta}_{\text{out}})$, $K, T > 0$, and $\theta_{\text{in}}(t) \equiv \theta_{\text{in}}$.

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -T^{-1}x_2(t) + KT^{-1}\sin(\theta_{\text{in}} - x_1(t))$$

Equilibria are $(\theta_{\text{in}} + n\pi, 0)$ since

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$
$$\dot{x}_2 = 0 \Rightarrow \sin(\theta_{in} - x_1) = 0 \Rightarrow x_1 = \theta_{\text{in}} + n\pi$$

Classification of Equilibria

Linearization gives the following characteristic equations:

n even:

$$\lambda^2 + T^{-1}\lambda + KT^{-1} = 0$$

$K > (4T)^{-1}$ gives stable focus
$0 < K < (4T)^{-1}$ gives stable node

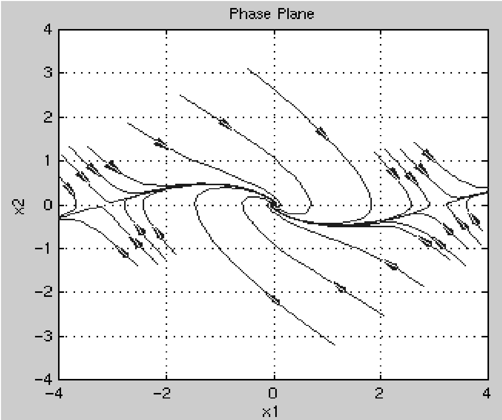
n odd:

$$\lambda^2 + T^{-1}\lambda - KT^{-1} = 0$$

Saddle points for all $K, T > 0$

Phase-Plane for PLL

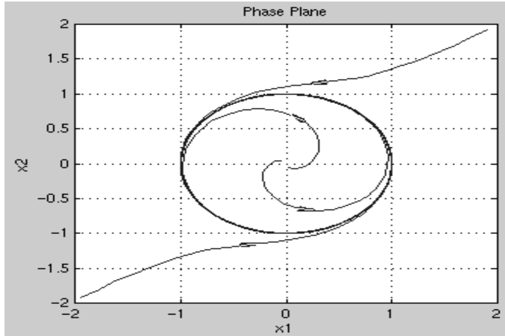
$(K,T) = (1/2, 1)$: focuses $(2k\pi, 0)$, saddle points $((2k+1)\pi, 0)$



Periodic Solutions

Example of an asymptotically stable periodic solution:

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2) \qquad (1)$$



Periodic solution: Polar coordinates.

$$x_1 = r\cos\theta \quad \Rightarrow \quad \dot{x}_1 = \cos\theta\dot{r} - r\sin\theta\dot{\theta}$$
$$x_2 = r\sin\theta \quad \Rightarrow \quad \dot{x}_2 = \sin\theta\dot{r} + r\cos\theta\dot{\theta}$$

implies

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Now, from (1)

$$\dot{x}_1 = r(1-r^2)\cos\theta - r\sin\theta$$
$$\dot{x}_2 = r(1-r^2)\sin\theta + r\cos\theta$$

gives

$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = 1$$

Only $r = 1$ is a stable equilibrium!

A system has a **periodic solution** if for some $T > 0$

$$x(t+T) = x(t), \qquad \forall t \geq 0$$

A **periodic orbit** is the image of x in the phase portrait.

- When does there exist a periodic solution?
- When is it stable?

Note that $x(t) \equiv$ const is by convention not regarded periodic

Flow

The solution of $\dot{x} = f(x)$ is sometimes denoted

$$\phi_t(x_0)$$

to emphasize the dependence on the initial point $x_0 \in \mathbb{R}^n$

$\phi_t(\cdot)$ is called the **flow**.

Poincaré Map

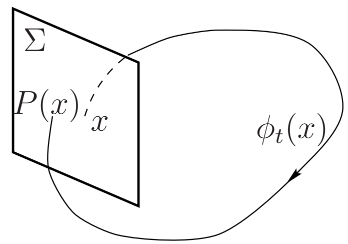
Assume $\phi_t(x_0)$ is a periodic solution with period T.

Let $\Sigma \subset \mathbb{R}^n$ be an $n-1$-dim hyperplane transverse to f at x_0.

Definition: The Poincaré map $P : \Sigma \to \Sigma$ is

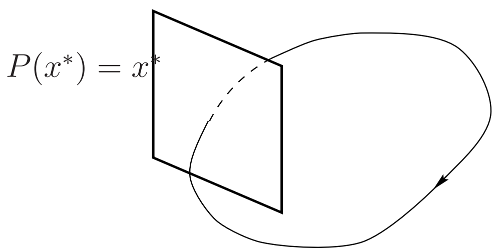
$$P(x) = \phi_{\tau(x)}(x)$$

where $\tau(x)$ is the time of first return.



Existence of Periodic Orbits

A point x^* such that $P(x^*) = x^*$ corresponds to a periodic orbit.



x^* is called a **fixed point** of P.

1 minute exercise: *What does a fixed point of P^k corresponds to?*

Stable Periodic Orbit

The linearization of P around x^* gives a matrix W such that

$$P(x) \approx Wx$$

if x is close to x^*.

- $\lambda_j(W) = 1$ for some j
- If $|\lambda_i(W)| < 1$ for all $i \neq j$, then the corresponding periodic orbit is **asymptotically stable**
- If $|\lambda_i(W)| > 1$ for some i, then the periodic orbit is **unstable**.

Example—Stable Unit Circle

Rewrite (1) in polar coordinates:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

Choose $\Sigma = \{(r, \theta) : r > 0, \theta = 2\pi k\}$.

The solution is

$$\phi_t(r_0, \theta_0) = \left([1 + (r_0^{-2} - 1)e^{-2t}]^{-1/2}, t + \theta_0 \right)$$

First return time from any point $(r_0, \theta_0) \in \Sigma$ is

$$\tau(r_0, \theta_0) = 2\pi.$$

The Poincaré map is

$$P(r_0, \theta_0) = \begin{pmatrix} [1 + (r_0^{-2} - 1)e^{-2\cdot 2\pi}]^{-1/2} \\ \theta_0 + 2\pi \end{pmatrix}$$

$(r_0, \theta_0) = (1, 2\pi k)$ is a fixed point.

The periodic solution that corresponds to $(r(t), \theta(t)) = (1, t)$ is asymptotically stable because

$$W = \frac{dP}{d(r_0, \theta_0)}(1, 2\pi k) = \begin{pmatrix} e^{-4\pi} & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ Stable periodic orbit (as we already knew for this example) !

EL2620 Nonlinear Control

Lecture 4

- Lyapunov methods for stability analysis

Today's Goal

You should be able to

- Prove local and global stability of equilibria using Lyapunov's method
- Prove stability of a set (e.g., a periodic orbit) using LaSalle's invariant set theorem

Alexandr Mihailovich Lyapunov (1857–1918)

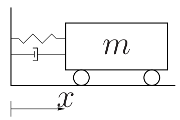
Master thesis "On the stability of ellipsoidal forms of equilibrium of rotating fluids," St. Petersburg University, 1884.

Doctoral thesis "The general problem of the stability of motion," 1892.

Formalized the idea:

If the total energy is dissipated, the system must be stable.

A Motivating Example



- Balance of forces yields

$$m\ddot{x} = -\underbrace{b\dot{x}|\dot{x}|}_{\text{damping}} - \underbrace{k_0 x - k_1 x^3}_{\text{spring}}, \qquad b, k_0, k_1 > 0$$

- Total energy = kinetic + potential energy: $V = \frac{mv^2}{2} + \int_0^x F_{spring} ds$

$$V(x, \dot{x}) = m\dot{x}^2/2 + k_0 x^2/2 + k_1 x^4/4 > 0, \qquad V(0,0) = 0$$

- Change in energy along any solution $x(t)$

$$\frac{d}{dt}V(x,\dot{x}) = m\dot{x}\ddot{x} + k_0 x\dot{x} + k_1 x^3 \dot{x} = -b|\dot{x}|^3 < 0, \quad \dot{x} \neq 0$$

Stability Definitions

Recall from Lecture 3 that an equilibrium $x = 0$ of $\dot{x} = f(x)$ is

Locally stable, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq 0$$

Locally asymptotically stable, if locally stable and

$$\|x(0)\| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0$$

Globally asymptotically stable, if asymptotically $\forall x(0) \in \mathbb{R}^n$.

Lyapunov Stability Theorem

Theorem: Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbb{R}^n$. If there exists a $\mathbb{C}^1$ function $V : \Omega \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \in \Omega$, $x \neq 0$

(3) $\dot{V}(x) \leq 0$ for all $x \in \Omega$

then $x = 0$ is locally stable. Furthermore, if

(4) $\dot{V}(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

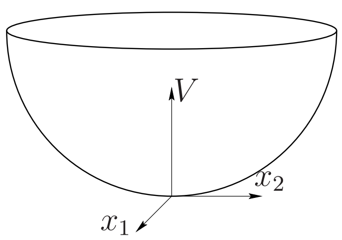
The result is called *Lyapunov's Direct Method*

Lyapunov Function

A function V that fulfills (1)–(3) is called a *Lyapunov function*.

Condition (3) means that V is non-increasing along all trajectories in Ω:

$$\dot{V}(x) = \frac{d}{dt}V(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) \leq 0$$



Conservation and Dissipation

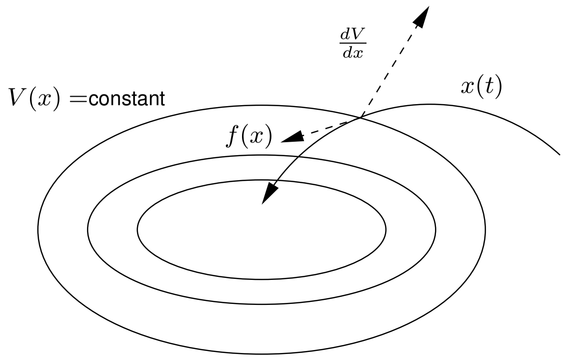
Conservation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x}f(x) = 0$, i.e. the vector field $f(x)$ is everywhere orthogonal to the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$.

Example: Total energy of a lossless mechanical system or total fluid in a closed system.

Dissipation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x}f(x) \leq 0$, i.e. the vector field $f(x)$ and the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$ make an obtuse angle.

Example: Total energy of a mechanical system with damping or total fluid in a system that leaks.

Geometric interpretation



Vector field points into sublevel sets

Trajectories can only go to lower values of $V(x)$

Boundedness:

For an trajectory $x(t)$

$$V((x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0))$$

which means that the whole trajectory lies in the set

$$\{z \mid V(z) \leq V(x(0))\}$$

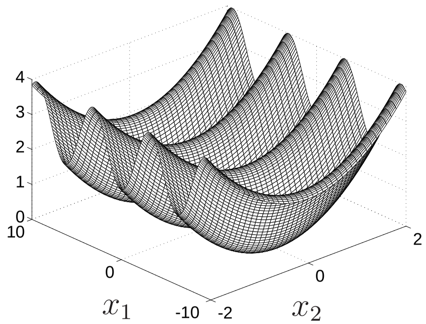
For stability it is thus important that the sublevel sets $\{z \mid V(z) \leq c)\}$ are locally bounded.

Example—Pendulum

Is the origin stable for a mathematical pendulum?

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Lyapunov function candidate: $V(x) = (1 - \cos x_1)g/\ell + x_2^2/2$



(1) $V(0) = 0$

(2) $V(x) > 0$ for $-2\pi < x_1 < 2\pi$ and $(x_1, x_2) \neq 0$

(3) $\dot{V}(x) = \frac{g}{\ell}\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = 0, \forall x$

Hence, $x = 0$ is locally stable.

Note that $x = 0$ is not asymptotically stable, so, of course, (4) is not fulfilled: $\dot{V}(x) \nless 0, \forall x \neq 0$.

Conservation of energy!

5 minute exercise: *Consider Example 2 from Lecture 3:*

$$\dot{x}_1 = x_2(t)$$
$$\dot{x}_2 = -x_1(t) - \epsilon x_1^2(t) x_2(t)$$

For what values of ϵ is the steady-state $(0,0)$ locally stable? Hint: try the "standard" Lyapunov function

$$V(x) = x^T x$$

Can you say something about global stability of the equilibrium?

Lyapunov Theorem for Global Asymptotic Stability

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $\mathbb{C}^1$ function $V : R^n \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \neq 0$

(3) $\dot{V}(x) < 0$ for all $x \neq 0$

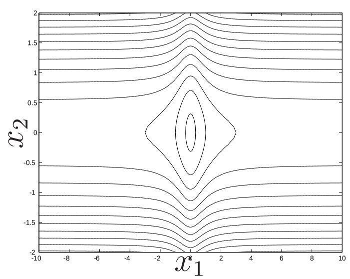
(4) $V(x) \to \infty$ as $\|x\| \to \infty$

then $x = 0$ is globally asymptotically stable.

Radial Unboundedness is Necessary

If (4) is not fulfilled, then global stability cannot be guaranteed.

Example: Assume $V(x) = x_1^2/(1+x_1^2) + x_2^2$ is a Lyapunov function for some system. Then might $x(t) \to \infty$ even if $\dot{V}(x) < 0$, as shown by the contour plot of $V(x)$:



Somewhat Stronger Assumptions

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $\mathbb{C}^1$ function $V : R^n \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \neq 0$

(3) $\dot{V}(x) \leq -\alpha V(x)$ for all x

(4) The sublevel sets $\{x | V(x) \leq c\}$ are bounded for all $c \geq 0$

then $x = 0$ is globally asymptotically stable.

Proof Idea

Assume $x(t) \neq 0$ (otherwise we have $x(\tau) = 0$ for all $\tau > t$). Then

$$\frac{\dot{V}(x)}{V(x)} \leq -\alpha$$

Integrating from 0 to t gives

$$\log V(x(t)) - \log V(x(0) \leq -\alpha t \;\Rightarrow\; V(x(t)) \leq e^{-\alpha t} V(x(0))$$

Hence, $V(x(t)) \to 0, t \to \infty$.

Using the properties of V it follows that $x(t) \to 0, t \to \infty$.

Converse Lyapunov theorems

Example: If the system is globally exponentially stable

$$||x(t)|| \leq M e^{-\beta t} ||x(0)||, \quad M > 0,\ \beta > 0$$

then there is a Lyapunov function that proves that it is globally asymptotically stable.

There exist also Lyapunov instability theorems!

Positive Definite Matrices

Definition: A matrix M is **positive definite** if $x^T M x > 0$ for all $x \neq 0$. It is **positive semidefinite** if $x^T M x \geq 0$ for all x.

Lemma:

- $M = M^T$ is positive definite $\Longleftrightarrow \lambda_i(M) > 0, \forall i$
- $M = M^T$ is positive semidefinite $\Longleftrightarrow \lambda_i(M) \geq 0, \forall i$

Note that if $M = M^T$ is positive definite, then the Lyapunov function candidate $V(x) = x^T M x$ fulfills $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$.

Symmetric Matrices

Assume that $M = M^T$. Then

$$\lambda_{\min}(M)\|x\|^2 \leq x^T M x \leq \lambda_{\max}(M)\|x\|^2$$

Hint: Use the factorization $M = U \Lambda U^T$, where U is an orthogonal matrix ($UU^T = I$) and $\Lambda =$ diag$(\lambda_1, \ldots, \lambda_n)$.

Lyapunov Stability for Linear Systems

Linear system: $\dot{x} = Ax$

Lyapunov equation: Consider the quadratic function

$$V(x) = x^T P x, \quad P = P^T > 0$$

$$\Rightarrow \quad \dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T \underbrace{(PA + A^T P)}_{Q} x = -x^T Q x$$

Thus, $\dot{V} < 0 \; \forall t$ if there exist a $Q = Q^T > 0$ such that

$$PA + A^T P = -Q$$

Global Asymptotic Stability: If Q is positive definite, then the Lyapunov Stability Theorem implies global asymptotic stability, and hence the eigenvalues of A must satisfy $\operatorname{Re} \lambda_i(A) < 0$ for all i

Converse Theorem for Linear Systems

If $\operatorname{Re} \lambda_i(A) < 0$, then for every symmetric positive definite Q there exist a symmetric positive definite matrix P such that

$$PA + A^T P = -Q$$

Proof: Choose $P = \int_0^\infty e^{A^T t} Q e^{At} dt$. Then

$$\begin{aligned} A^T P + PA &= \int_0^\infty \left(A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A \right) dt \\ &= \int_0^\infty \left(\frac{d}{dt} e^{A^T t} Q e^{At} \right) dt = \left[e^{A^T t} Q e^{At} \right]_0^\infty = -Q \end{aligned}$$

Interpretation

Assume $\dot{x} = Ax$, $x(0) = z$. Then

$$\int_0^\infty x^T(t) Q x(t) dt = z^t \left(\int_0^\infty e^{A^T t} Q e^{At} dt \right) z = z^t P z$$

Thus $v(z) = z^T P z$ is the cost-to-go from the point z (no input) with integral quadratic cost function using weighting matrix Q.

Lyapunov's Linearization Method

Recall from Lecture 3:

Theorem: Let x_0 be an equilibrium of $\dot{x} = f(x)$ with $f \in \mathbb{C}^1$. Denote $A = \frac{\partial f}{\partial x}(x_0)$ and $\alpha(A) = \max \operatorname{Re}(\lambda(A))$.

(1) If $\alpha(A) < 0$ then x_0 is asymptotically stable

(2) If $\alpha(A) > 0$ then x_0 is unstable

Proof of (1) in Lyapunov's Linearization

Let $f(x) = Ax + g(x)$ where $\lim_{\|x\|\to 0} \|g(x)\|/\|x\| = 0$. The Lyapunov function candidate $V(x) = x^T P x$ satisfies $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, and

$$\begin{aligned}\dot V(x) &= x^T P f(x) + f^T(x) P x \\ &= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)] P x \\ &= x^T (PA + A^T P) x + 2x^T P g(x) \\ &= -x^T Q x + 2x^T P g(x)\end{aligned}$$

where

$$x^T Q x \geq \lambda_{\min}(Q)\|x\|^2$$

- we need to show that $\|2x^T P g(x)\| < \lambda_{min}(Q)\|x\|^2$

For all $\gamma > 0$ there exists $r > 0$ such that

$$\|g(x)\| < \gamma\|x\|, \qquad \forall \|x\| < r$$

Thus,

$$\dot V < -\lambda_{min}(Q)\|x\|^2 + 2\gamma\lambda_{max}(P)\|x\|^2$$

which becomes strictly negative if we choose

$$\gamma < \frac{1}{2}\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$$

LaSalle's Theorem for Global Asymptotic Stability

Theorem: Let $\dot x = f(x)$ and $f(0) = 0$. If there exists a $\mathbb{C}^1$ function $V: R^n \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \neq 0$

(3) $\dot V(x) \leq 0$ for all x

(4) $V(x) \to \infty$ as $\|x\| \to \infty$

(5) The only solution of $\dot x = f(x)$ such that $\dot V(x) = 0$ is $x(t) = 0$ for all t

then $x = 0$ is globally asymptotically stable.

A Motivating Example (cont'd)

$$\begin{aligned}& m\ddot x = -b\dot x|\dot x| - k_0 x - k_1 x^3 \\ & V(x) = (2m\dot x^2 + 2k_0 x^2 + k_1 x^4)/4 > 0, \qquad V(0,0) = 0 \\ & \dot V(x) = -b|\dot x|^3\end{aligned}$$

Assume that there is a trajectory with $\dot x(t) = 0$, $x(t) \neq 0$. Then

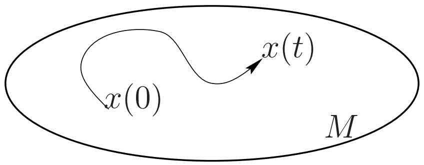
$$\frac{d}{dt}\dot x(t) = -\frac{k_0}{m}x(t) - \frac{k_1}{m}x^3(t) \neq 0,$$

which means that $\dot x(t)$ can not stay constant.

Hence, $x(t) = 0$ is the only possible trajectory for which $\dot V(x) = 0$, and the LaSalle theorem gives global asymptotic stability.

Invariant Sets

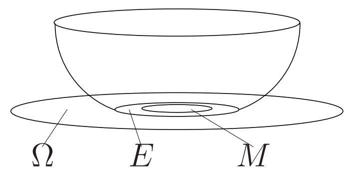
Definition: A set M is **invariant** with respect to $\dot{x} = f(x)$, if $x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.

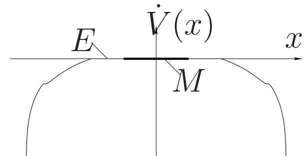


Definition: $x(t)$ **approaches** a set M as $t \to \infty$, if for each $\epsilon > 0$ there is a $T > 0$ such that $\mathrm{dist}(x(t), M) < \epsilon$ for all $t > T$. Here $\mathrm{dist}(p, M) = \inf_{x \in M} \|p - x\|$.

LaSalle's Invariant Set Theorem

Theorem: Let $\Omega \subset \mathbb{R}^n$ be a compact set invariant with respect to $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that $\dot{V}(x) \leq 0$ for $x \in \Omega$. Let E be the set of points in Ω where $\dot{V}(x) = 0$. If M is the largest invariant set in E, then every solution with $x(0) \in \Omega$ approaches M as $t \to \infty$.





Note that V does **not** have to be a positive definite function.

Example—Periodic Orbit

Show that $x(t)$ approaches $\{x : \|x\| = 1\} \cup \{0\}$ for

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

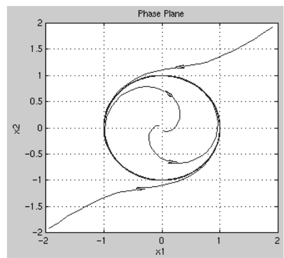
It is possible to show that $\Omega = \{\|x\| \leq R\}$ is invariant for sufficiently large $R > 0$. Let $V(x) = (x_1^2 + x_2^2 - 1)^2$.

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = 2(x_1^2 + x_2^2 - 1)\frac{d}{dt}(x_1^2 + x_2^2 - 1)$$
$$= -2(x_1^2 + x_2^2 - 1)^2(x_1^2 + x_2^2) \leq 0, \quad \forall x \in \Omega$$

$$E = \{x \in \Omega : \dot{V}(x) = 0\} = \{x : \|x\| = 1\} \cup \{0\}$$

The largest invariant set of E is $M = E$ because

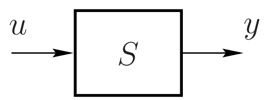
$$\frac{d}{dt}(x_1^2 + x_2^2 - 1) = -2(x_1^2 + x_2^2 - 1)(x_1^2 + x_2^2) = 0 \quad \text{for } x \in M$$



EL2620 Nonlinear Control

Lecture 5

- Input–output stability

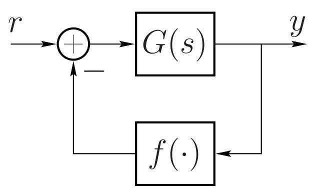


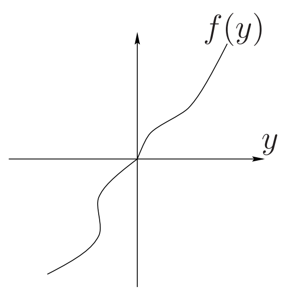
Today's Goal

You should be able to

- derive the gain of a system
- analyze stability using
  - Small Gain Theorem
  - Circle Criterion
  - Passivity

History



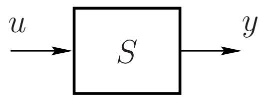


For what $G(s)$ and $f(\cdot)$ is the closed-loop system stable?

- Luré and Postnikov's problem (1944)
- Aizerman's conjecture (1949) (False!)
- Kalman's conjecture (1957) (False!)
- Solution by Popov (1960) (Led to the Circle Criterion)

Gain

Idea: Generalize the concept of gain to nonlinear dynamical systems



The **gain** γ of S is **the largest amplification** from u to y

Here S can be a constant, a matrix, a linear time-invariant system, etc

Question: How should we measure the size of u and y?

Norms

A norm $\|\cdot\|$ measures size

Definition:
A norm is a function $\|\cdot\| : \Omega \to \mathbb{R}^+$, such that for all $x, y \in \Omega$

- $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|\alpha x\| = |\alpha| \cdot \|x\|$, for all $\alpha \in \mathbb{R}$

Examples:

Euclidean norm: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$

Max norm: $\|x\| = \max\{|x_1|, \ldots, |x_n|\}$

Gain of a Matrix

Every matrix $M \in \mathbf{C}^{n \times n}$ has a **singular value decomposition**

$$M = U \Sigma V^*$$

where

$$\Sigma = \text{diag}\{\sigma_1, \ldots, \sigma_n\}\,; \quad U^*U = I\,; \quad V^*V = I$$

σ_i - the singular values

The "gain" of M is the largest singular value of M:

$$\sigma_{\max}(M) = \sigma_1 = \sup_{x \in \mathbb{R}^n} \frac{\|Mx\|}{\|x\|}$$

where $\|\cdot\|$ is the Euclidean norm.

Eigenvalues are not gains

The spectral radius of a matrix M

$$\rho(M) = \max_i |\lambda_i(M)|$$

is **not a gain** (nor a norm).

Why? *What amplification is described by the eigenvalues?*

Signal Norms

A signal x is a function $x : \mathbb{R}^+ \to \mathbb{R}$.

A signal norm $\|\cdot\|_k$ is a norm on the space of signals x.

Examples:

2-norm (energy norm): $\|x\|_2 = \sqrt{\int_0^\infty |x(t)|^2 dt}$

sup-norm: $\|x\|_\infty = \sup_{t \in \mathbb{R}^+} |x(t)|$

Parseval's Theorem

$\mathcal{L}_2$ denotes the space of signals with bounded energy: $\|x\|_2 < \infty$

Theorem: If $x, y \in \mathcal{L}_2$ have the Fourier transforms

$$X(i\omega) = \int_0^\infty e^{-i\omega t} x(t)dt, \qquad Y(i\omega) = \int_0^\infty e^{-i\omega t} y(t)dt,$$

then

$$\int_0^\infty y(t)x(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty Y^*(i\omega)X(i\omega)d\omega.$$

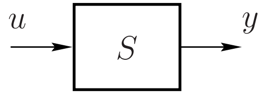
In particular,

$$\|x\|_2^2 = \int_0^\infty |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty |X(i\omega)|^2 d\omega.$$

The power calculated in the time domain equals the power calculated in the frequency domain

System Gain

A system S is a map from $\mathcal{L}_2$ to $\mathcal{L}_2$: $y = S(u)$.

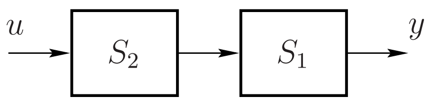


The gain of S is defined as $\gamma(S) = \sup_{u\in\mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u\in\mathcal{L}_2} \frac{\|S(u)\|_2}{\|u\|_2}$

Example: The gain of a scalar static system $y(t) = \alpha u(t)$ is

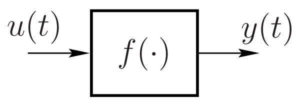
$$\gamma(\alpha) = \sup_{u\in\mathcal{L}_2} \frac{\|\alpha u\|_2}{\|u\|_2} = \sup_{u\in\mathcal{L}_2} \frac{|\alpha|\|u\|_2}{\|u\|_2} = |\alpha|$$

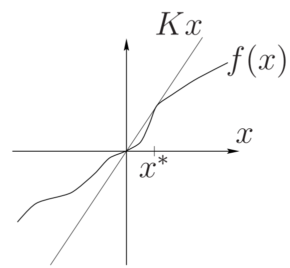
2 minute exercise: *Show that* $\gamma(S_1 S_2) \le \gamma(S_1)\gamma(S_2)$.



Gain of a Static Nonlinearity

Lemma: A static nonlinearity f such that $|f(x)| \le K|x|$ and $f(x^*) = Kx^*$ has gain $\gamma(f) = K$.



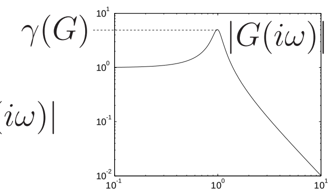


Proof: $\|y\|_2^2 = \int_0^\infty f^2\big(u(t)\big)dt \le \int_0^\infty K^2 u^2(t)dt = K^2\|u\|_2^2$, where $u(t) = x^*$, $t \in (0,1)$, gives equality, so

$$\gamma(f) = \sup_{u\in\mathcal{L}_2} \|y\|_2/\|u\|_2 = K$$

Gain of a Stable Linear System

Lemma:



$$\gamma(G) = \sup_{u\in\mathcal{L}_2} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\omega\in(0,\infty)} |G(i\omega)|$$

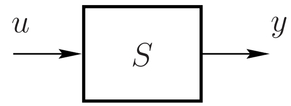
Proof: Assume $|G(i\omega)| \le K$ for $\omega \in (0,\infty)$ and $|G(i\omega^*)| = K$ for some ω^*. Parseval's theorem gives

$$\|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|^2 |U(i\omega)|^2 d\omega \le K^2 \|u\|_2^2$$

Arbitrary close to equality by choosing $u(t)$ close to $\sin \omega^* t$.

BIBO Stability

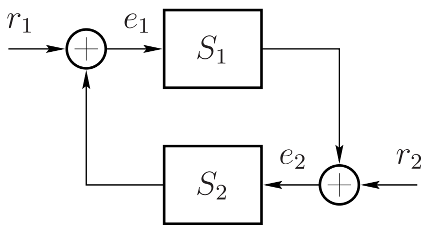
Definition:
S is bounded-input bounded-output (BIBO) stable if $\gamma(S) < \infty$.



$$\gamma(S) = \sup_{u\in\mathcal{L}_2} \frac{\|S(u)\|_2}{\|u\|_2}$$

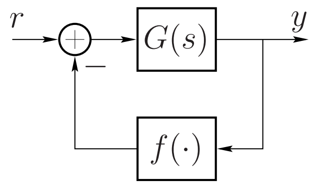
Example: If $\dot{x} = Ax$ is asymptotically stable then $G(s) = C(sI - A)^{-1}B + D$ is BIBO stable.

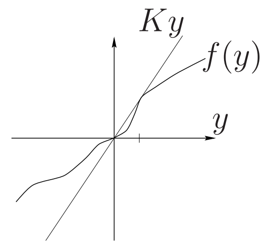
The Small Gain Theorem



Theorem: Assume S_1 and S_2 are BIBO stable. If $\gamma(S_1)\gamma(S_2) < 1$, then the closed-loop system is BIBO stable from (r_1, r_2) to (e_1, e_2)

Example—Static Nonlinear Feedback





$$G(s) = \frac{2}{(s+1)^2}, \qquad 0 \le \frac{f(y)}{y} \le K, \quad \forall y \ne 0, \quad f(0) = 0$$

$\gamma(G) = 2$ and $\gamma(f) \le K$.

Small Gain Theorem gives BIBO stability for $K \in (0, 1/2)$.

"Proof" of the Small Gain Theorem

$$\|e_1\|_2 \le \|r_1\|_2 + \gamma(S_2)[\|r_2\|_2 + \gamma(S_1)\|e_1\|_2]$$

gives

$$\|e_1\|_2 \le \frac{\|r_1\|_2 + \gamma(S_2)\|r_2\|_2}{1-\gamma(S_2)\gamma(S_1)}$$

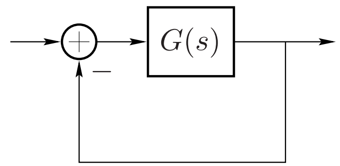
$\gamma(S_2)\gamma(S_1) < 1$, $\|r_1\|_2 < \infty$, $\|r_2\|_2 < \infty$ give $\|e_1\|_2 < \infty$.
Similarly we get

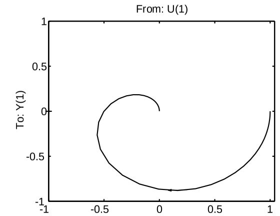
$$\|e_2\|_2 \le \frac{\|r_2\|_2 + \gamma(S_1)\|r_1\|_2}{1-\gamma(S_1)\gamma(S_2)}$$

so also e_2 is bounded.

Note: Formal proof requires $\|\cdot\|_{2e}$, see Khalil

The Nyquist Theorem

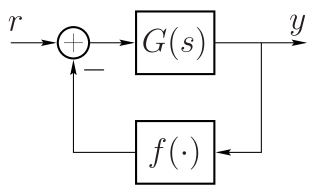


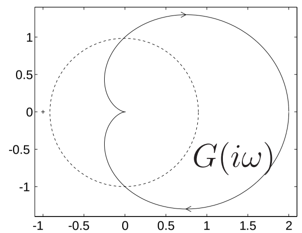


Theorem: If G has no poles in the right half plane and the Nyquist curve $G(i\omega)$, $\omega \in [0,\infty)$, does not encircle -1, then the closed-loop system is stable.

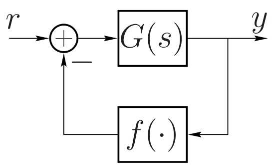
Small Gain Theorem can be Conservative

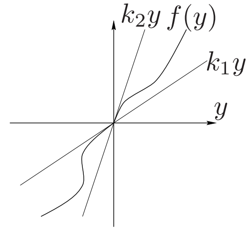
Let $f(y) = Ky$ in the previous example. Then the Nyquist Theorem proves stability for all $K \in [0,\infty)$, while the Small Gain Theorem only proves stability for $K \in (0, 1/2)$.

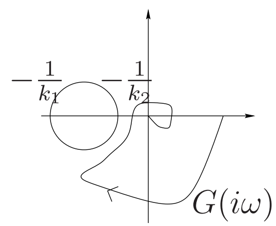




The Circle Criterion





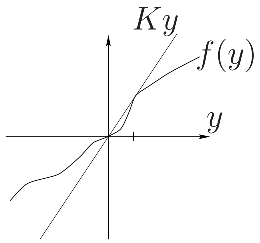


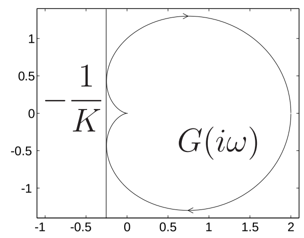
Theorem: Assume that $G(s)$ has no poles in the right half plane, and

$$0 \le k_1 \le \frac{f(y)}{y} \le k_2, \quad \forall y \ne 0, \qquad f(0) = 0$$

If the Nyquist curve of $G(s)$ does not encircle or intersect the circle defined by the points $-1/k_1$ and $-1/k_2$, then the closed-loop system is BIBO stable.

Example—Static Nonlinear Feedback (cont'd)





The "circle" is defined by $-1/k_1 = -\infty$ and $-1/k_2 = -1/K$. Since

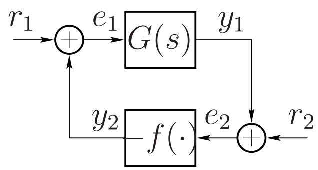
$$\min \text{Re}\, G(i\omega) = -1/4$$

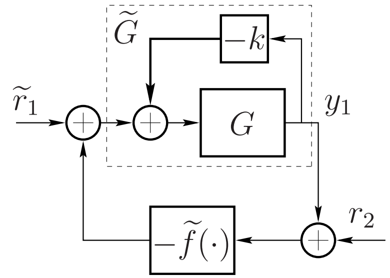
the Circle Criterion gives that the system is BIBO stable if $K \in (0, 4)$.

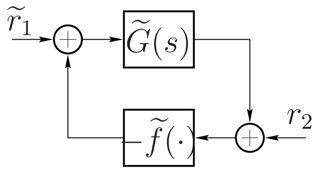
Proof of the Circle Criterion

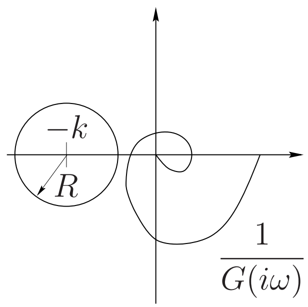
Let $k = (k_1 + k_2)/2$, $\widetilde{f}(y) = f(y) - ky$, and $\widetilde{r}_1 = r_1 - kr_2$:

$$\left|\frac{\widetilde{f}(y)}{y}\right| \le \frac{k_2 - k_1}{2} =: R, \quad \forall y \ne 0, \qquad \widetilde{f}(0) = 0$$





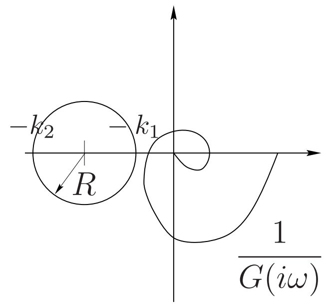


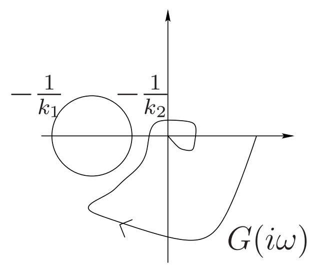


Small Gain Theorem gives stability if $|\widetilde{G}(i\omega)|R < 1$, where $\widetilde{G} = \dfrac{G}{1 + kG}$ is stable (This has to be checked later). Hence,

$$\frac{1}{|\widetilde{G}(i\omega)|} = \left|\frac{1}{G(i\omega)} + k\right| > R$$

The curve $G^{-1}(i\omega)$ and the circle $\{z \in \mathbf{C} : |z + k| > R\}$ mapped through $z \mapsto 1/z$ gives the result:





Note that $\dfrac{G}{1 + kG}$ is stable since $-1/k$ is inside the circle.

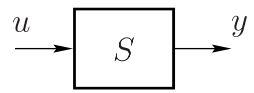
Note that $G(s)$ may have poles on the imaginary axis, e.g., integrators are allowed

Passivity and BIBO Stability

The main result: Feedback interconnections of passive systems are passive, and BIBO stable (under some additional mild criteria)

Scalar Product

Scalar product for signals y and u



$$\langle y, u \rangle_T = \int_0^T y^T(t)u(t)dt$$

If u and y are interpreted as vectors then $\langle y, u \rangle_T = |y|_T |u|_T \cos\phi$

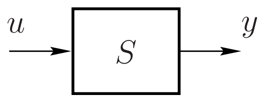
$|y|_T = \sqrt{\langle y, y \rangle_T}$ - length of y, ϕ - angle between u and y

Cauchy-Schwarz Inequality: $\langle y, u \rangle_T \leq |y|_T |u|_T$

Example: $u = \sin t$ and $y = \cos t$ are orthogonal if $T = k\pi$, because

$$\cos\phi = \frac{\langle y, u \rangle_T}{|y|_T |u|_T} = 0$$

Passive System



Definition: Consider signals $u, y : [0, T] \to \mathbb{R}^m$. The system S is **passive** if

$$\langle y, u \rangle_T \geq 0, \quad \text{for all } T > 0 \text{ and all } u$$

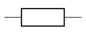
and **strictly passive** if there exists $\epsilon > 0$ such that

$$\langle y, u \rangle_T \geq \epsilon(|y|_T^2 + |u|_T^2), \quad \text{for all } T > 0 \text{ and all } u$$

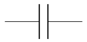
Warning: There exist many other definitions for strictly passive

2 minute exercise: Is the pure delay system $y(t) = u(t-\theta)$ passive? Consider for instance the input $u(t) = sin\left((\pi/\theta)t\right)$.

Example—Passive Electrical Components



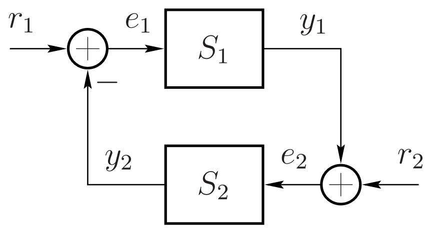
$$u(t) = Ri(t) : \langle u, i\rangle_T = \int_0^T Ri^2(t)dt \ge R\langle i, i\rangle_T \ge 0$$



$$i = C\frac{du}{dt} : \langle u, i\rangle_T = \int_0^T u(t)C\frac{du}{dt}dt = \frac{Cu^2(T)}{2} \ge 0$$

$$u = L\frac{di}{dt} : \langle u, i\rangle_T = \int_0^T L\frac{di}{dt}i(t)dt = \frac{Li^2(T)}{2} \ge 0$$

Feedback of Passive Systems is Passive



Lemma: If S_1 and S_2 are passive then the closed-loop system from (r_1, r_2) to (y_1, y_2) is also passive.

Proof:

$$\langle y, e\rangle_T = \langle y_1, e_1\rangle_T + \langle y_2, e_2\rangle_T = \langle y_1, r_1 - y_2\rangle_T + \langle y_2, r_2 + y_1\rangle_T$$
$$= \langle y_1, r_1\rangle_T + \langle y_2, r_2\rangle_T = \langle y, r\rangle_T$$

Hence, $\langle y, r\rangle_T \ge 0$ if $\langle y_1, e_1\rangle_T \ge 0$ and $\langle y_2, e_2\rangle_T \ge 0$.

Passivity of Linear Systems

Theorem: An asymptotically stable linear system $G(s)$ is **passive** if and only if

$$\text{Re}\, G(i\omega) \ge 0, \qquad \forall \omega > 0$$

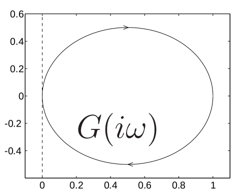
It is **strictly passive** if and only if there exists $\epsilon > 0$ such that

$$\text{Re}\, G(i\omega - \epsilon) \ge 0, \qquad \forall \omega > 0$$

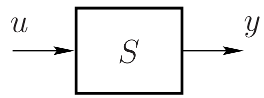
Example:

$G(s) = \dfrac{1}{s+1}$ is strictly passive,

$G(s) = \dfrac{1}{s}$ is passive but **not** strictly passive.



A Strictly Passive System Has Finite Gain



Lemma: If S is strictly passive then $\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} < \infty$.

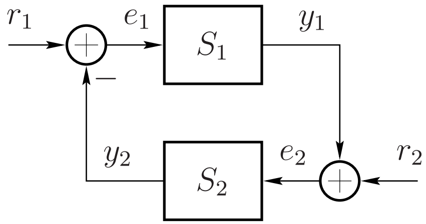
Proof:

$$\epsilon(|y|^2_\infty + |u|^2_\infty) \le \langle y, u\rangle_\infty \le |y|_\infty \cdot |u|_\infty = \|y\|_2 \cdot \|u\|_2$$

Hence, $\epsilon\|y\|_2^2 \le \|y\|_2 \cdot \|u\|_2$, so

$$\|y\|_2 \le \frac{1}{\epsilon}\|u\|_2$$

The Passivity Theorem



Theorem: If S_1 is strictly passive and S_2 is passive, then the closed-loop system is BIBO stable from r to y.

Proof

S_1 strictly passive and S_2 passive give

$$\epsilon\left(|y_1|_T^2 + |e_1|_T^2\right) \le \langle y_1, e_1\rangle_T + \langle y_2, e_2\rangle_T = \langle y, r\rangle_T$$

Therefore

$$|y_1|_T^2 + \langle r_1 - y_2, r_1 - y_2\rangle_T \le \frac{1}{\epsilon}\langle y, r\rangle_T$$

or

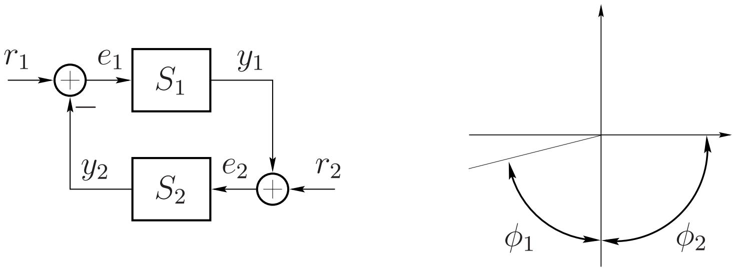
$$|y_1|_T^2 + |y_2|_T^2 - 2\langle y_2, r_2\rangle_T + |r_1|_T^2 \le \frac{1}{\epsilon}\langle y, r\rangle_T$$

Hence

$$|y|_T^2 \le 2\langle y_2, r_2\rangle_T + \frac{1}{\epsilon}\langle y, r\rangle_T \le \left(2 + \frac{1}{\epsilon}\right)|y|_T|r|_T$$

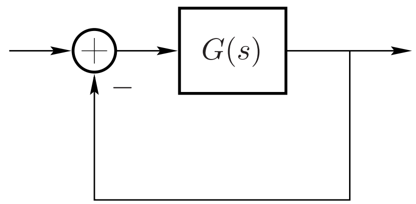
Let $T \to \infty$ and the result follows.

The Passivity Theorem is a "Small Phase Theorem"



S_2 passive $\quad\Rightarrow\quad \cos\phi_2 \ge 0 \quad\Rightarrow\quad |\phi_2| \le \pi/2$

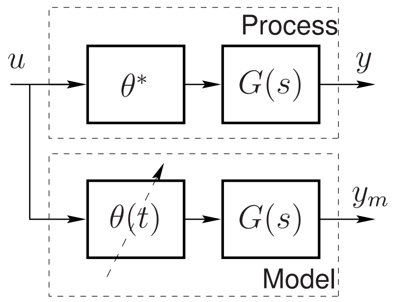
S_1 strictly passive $\quad\Rightarrow\quad \cos\phi_1 > 0 \quad\Rightarrow\quad |\phi_1| < \pi/2$



2 minute exercise: *Apply the Passivity Theorem and compare it with the Nyquist Theorem. What about conservativeness? [Compare the discussion on the Small Gain Theorem.]*

Example—Gain Adaptation

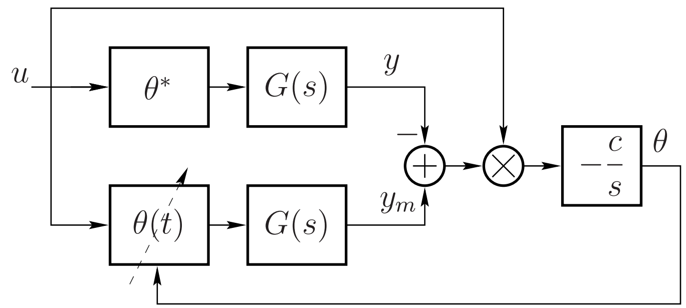
Applications in telecommunication channel estimation and in noise cancellation etc.



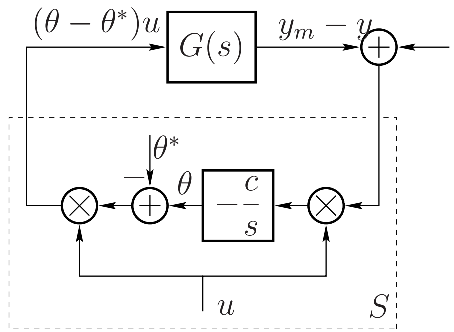
Adaptation law:

$$\frac{d\theta}{dt} = -cu(t)[y_m(t) - y(t)], \qquad c > 0.$$

Gain Adaptation—Closed-Loop System



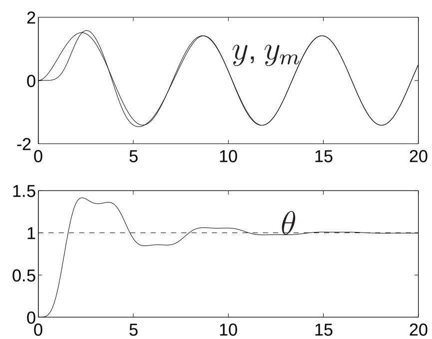
Gain Adaptation is BIBO Stable



S is **passive** (see exercises).
If $G(s)$ is **strictly passive**, the closed-loop system is BIBO stable

Simulation of Gain Adaptation

Let $G(s) = \dfrac{1}{s+1}$, $c = 1$, $u = \sin t$, and $\theta(0) = 0$.



Storage Function

Consider the nonlinear control system

$$\dot{x} = f(x,u), \qquad y = h(x)$$

A **storage function** is a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- $V(0) = 0$ and $V(x) \geq 0, \quad \forall x \neq 0$
- $\dot{V}(x) \leq u^T y, \quad \forall x, u$

Remark:

- $V(T)$ represents the stored energy in the system
- $\underbrace{V(x(T))}_{\text{stored energy at } t=T} \leq \underbrace{\int_0^T y(t)u(t)dt}_{\text{absorbed energy}} + \underbrace{V(x(0))}_{\text{stored energy at } t=0} \quad , \forall T > 0$

Storage Function and Passivity

Lemma: If there exists a storage function V for a system

$$\dot{x} = f(x,u), \qquad y = h(x)$$

with $x(0) = 0$, then the system is passive.

Proof: For all $T > 0$,

$$\langle y, u\rangle_T = \int_0^T y(t)u(t)dt \geq V(x(T)) - V(x(0)) = V(x(T)) \geq 0$$

Lyapunov vs. Passivity

Storage function is a generalization of Lyapunov function

Lyapunov idea: "Energy is decreasing"

$$\dot{V} \leq 0$$

Passivity idea: "Increase in stored energy $\leq$ Added energy"

$$\dot{V} \leq u^T y$$

Example—KYP Lemma

Consider an asymptotically stable linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

Assume there exists positive definite matrices P, Q such that

$$A^T P + PA = -Q, \qquad B^T P = C$$

Consider $V = 0.5x^T P x$. Then

$$\dot{V} = 0.5(\dot{x}^T P x + x^T P \dot{x}) = 0.5x^T(A^T P + PA)x + uB^T P x$$
$$= -0.5x^T Q x + uy < uy, \quad x \neq 0$$

and hence the system is strictly passive. This fact is part of the Kalman-Yakubovich-Popov lemma.

Today’s Goal

You should be able to

- derive the gain of a system
- analyze stability using
  - Small Gain Theorem
  - Circle Criterion
  - Passivity

EL2620 Nonlinear Control

Lecture 6

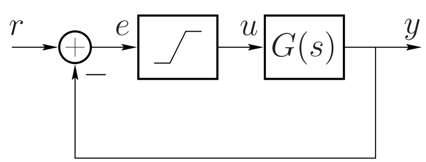
- Describing function analysis

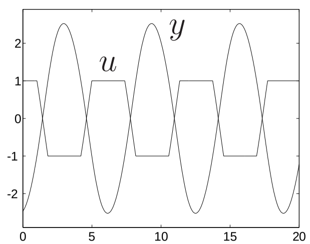
Today's Goal

You should be able to

- Derive describing functions for static nonlinearities
- Analyze existence and stability of periodic solutions by describing function analysis

Motivating Example





$G(s) = \dfrac{4}{s(s+1)^2}$ and $u =$ sat e give a stable oscillation.

- How can the oscillation be predicted?

A Frequency Response Approach

Nyquist / Bode:

A (linear) feedback system will have sustained oscillations (center) if the loop-gain is 1 at the frequency where the phase lag is -180^o

But, can we talk about the frequency response, in terms of gain and phase lag, of a static nonlinearity?

Fourier Series

A periodic function $u(t) = u(t+T)$ has a Fourier series expansion

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}\sin[n\omega t + \arctan(a_n/b_n)]$$

where $\omega = 2\pi/T$ and

$$a_n(\omega) = \frac{2}{T}\int_0^T u(t)\cos n\omega t\,dt, \quad b_n(\omega) = \frac{2}{T}\int_0^T u(t)\sin n\omega t\,dt$$

Note: Sometimes we make the change of variable $t \to \phi/\omega$

The Fourier Coefficients are Optimal

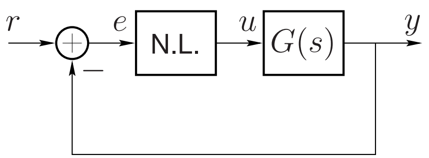
The finite expansion

$$\widehat{u}_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k}(a_n \cos n\omega t + b_n \sin n\omega t)$$

solves

$$\min_{\hat{u}} \frac{2}{T}\int_0^T \left[u(t) - \widehat{u}_k(t)\right]^2 dt$$

Key Idea



$e(t) = A\sin\omega t$ gives

$$u(t) = \sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}\sin[n\omega t + \arctan(a_n/b_n)]$$

If $|G(in\omega)| \ll |G(i\omega)|$ for $n \geq 2$, then $n = 1$ suffices, so that

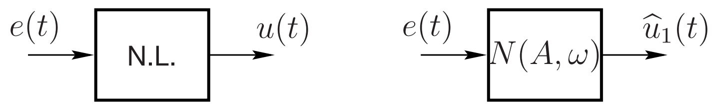
$$y(t) \approx |G(i\omega)|\sqrt{a_1^2 + b_1^2}\sin[\omega t + \arctan(a_1/b_1) + \arg G(i\omega)]$$

That is, we assume all higher harmonics are filtered out by G

Definition of Describing Function

The **describing function** is

$$N(A,\omega) = \frac{b_1(\omega) + ia_1(\omega)}{A}$$



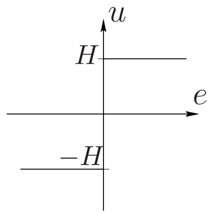
If G is low pass and $a_0 = 0$, then

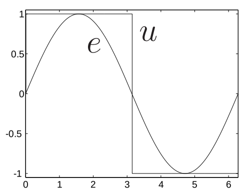
$$\widehat{u}_1(t) = |N(A,\omega)|A\sin[\omega t + \arg N(A,\omega)]$$

can be used instead of $u(t)$ to analyze the system.

Amplitude dependent gain and phase shift!

Describing Function for a Relay





$\phi = 2\pi t/T$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi\, d\phi = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi\, d\phi = \frac{2}{\pi}\int_0^{\pi} H\sin\phi\, d\phi = \frac{4H}{\pi}$$

The describing function for a relay is thus

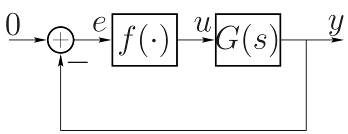
$$N(A) = \frac{b_1(\omega) + i a_1(\omega)}{A} = \frac{4H}{\pi A}$$

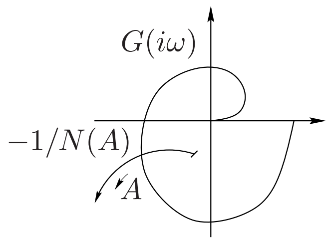
Odd Static Nonlinearities

Assume $f(\cdot)$ and $g(\cdot)$ are odd (i.e. $f(-e) = -f(e)$) static nonlinearities with describing functions N_f and N_g. Then,

- $\operatorname{Im} N_f(A,\omega) = 0$
- $N_f(A,\omega) = N_f(A)$
- $N_{\alpha f}(A) = \alpha N_f(A)$
- $N_{f+g}(A) = N_f(A) + N_g(A)$

Existence of Periodic Solutions



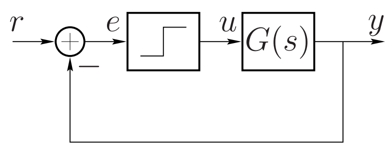


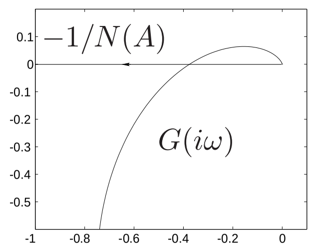
Proposal: sustained oscillations if loop-gain 1 *and phase-lag* -180^o

$$G(i\omega)N(A) = -1 \quad \Leftrightarrow \quad G(i\omega) = -1/N(A)$$

The intersections of the curves $G(i\omega)$ and $-1/N(A)$ give ω and A for a possible periodic solution.

Periodic Solutions in Relay System



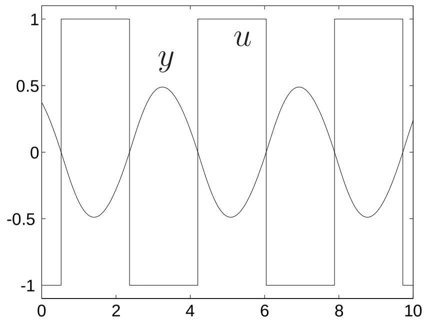


$$G(s) = \frac{3}{(s+1)^3} \quad \text{with feedback} \quad u = -\text{sgn}\, y$$

No phase lag in $f(\cdot)$, $\arg G(i\omega) = -\pi$ for $\omega = \sqrt{3} = 1.7$

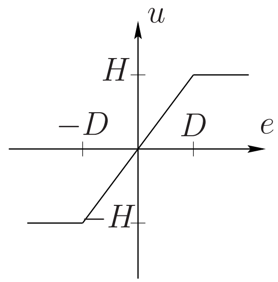
$$G(i\sqrt{3}) = -3/8 = -1/N(A) = -\pi A/4 \quad \Rightarrow \quad A = 12/8\pi \approx 0.48$$

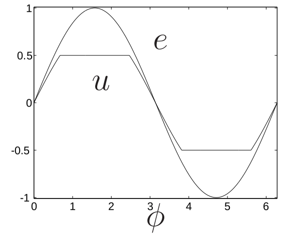
The prediction via the describing function agrees very well with the true oscillations:



Note that G filters out almost all higher-order harmonics.

Describing Function for a Saturation





Let $e(t) = A\sin\omega t = A\sin\phi$. First set $H = D$. Then for $\phi \in (0,\pi)$

$$u(\phi) = \begin{cases} A\sin\phi, & \phi \in (0,\phi_0)\cup(\pi-\phi_0,\pi) \\ D, & \phi \in (\phi_0, \pi-\phi_0) \end{cases}$$

where $\phi_0 = \arcsin D/A$.

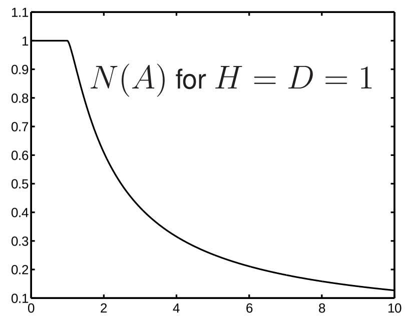
$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi\,d\phi = 0$$

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi\,d\phi = \frac{4}{\pi}\int_0^{\pi/2} u(\phi)\sin\phi\,d\phi \\ &= \frac{4A}{\pi}\int_0^{\phi_0}\sin^2\phi\,d\phi + \frac{4D}{\pi}\int_{\phi_0}^{\pi/2}\sin\phi\,d\phi \\ &= \frac{A}{\pi}\Big(2\phi_0 + \sin 2\phi_0\Big) \end{aligned}$$

Hence, if $H = D$, then $N(A) = \frac{1}{\pi}\Big(2\phi_0 + \sin 2\phi_0\Big)$.

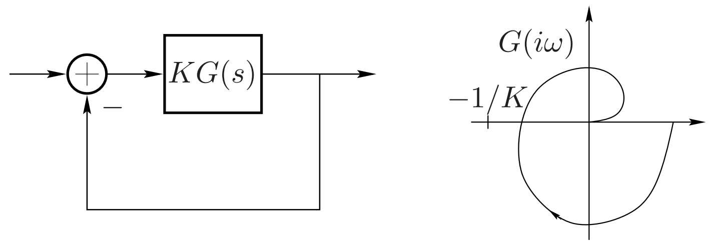
If $H \neq D$, then the rule $N_{\alpha f}(A) = \alpha N_f(A)$ gives

$$N(A) = \frac{H}{D\pi}\Big(2\phi_0 + \sin 2\phi_0\Big)$$



5 minute exercise: *What oscillation amplitude and frequency do the describing function analysis predict for the "Motivating Example"?*

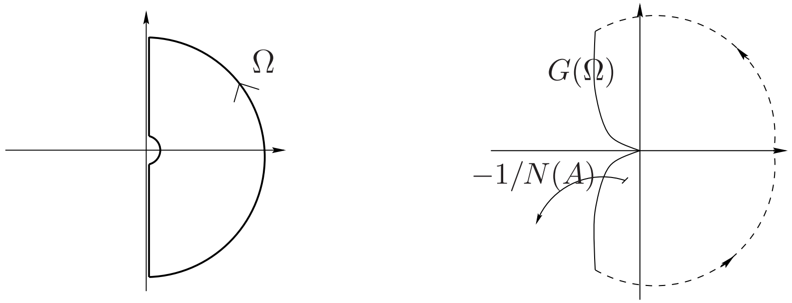
The Nyquist Theorem



Assume that G is stable, and K is a positive gain.

- If $G(i\omega)$ goes through the point $-1/K$ the closed-loop system displays sustained oscillations
- If $G(i\omega)$ encircles the point $-1/K$, then the closed-loop system is unstable (growing amplitude oscillations).
- If $G(i\omega)$ does not encircle the point $-1/K$, then the closed-loop system is stable (damped oscillations)

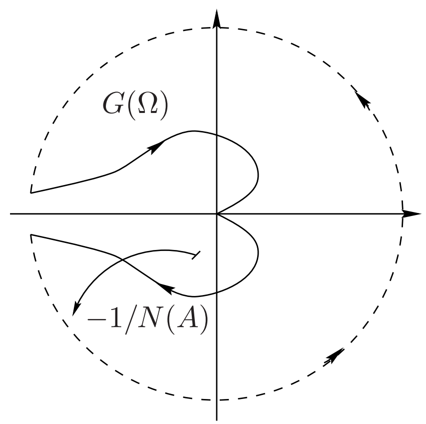
Stability of Periodic Solutions



Assume that $G(s)$ is stable.

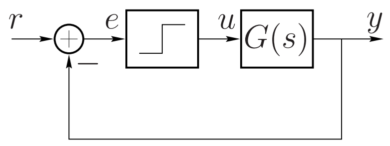
- If $G(\Omega)$ encircles the point $-1/N(A)$, then the oscillation amplitude is increasing.
- If $G(\Omega)$ does not encircle the point $-1/N(A)$, then the oscillation amplitude is decreasing.

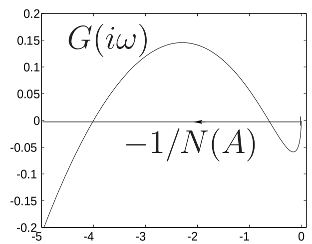
An Unstable Periodic Solution



An intersection with amplitude A_0 is unstable if $A < A_0$ leads to decreasing amplitude and $A > A_0$ leads to increasing.

Stable Periodic Solution in Relay System



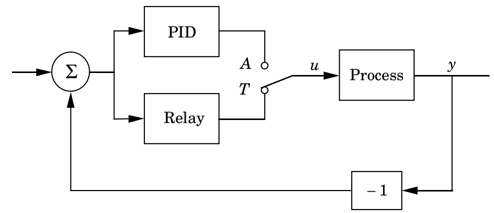


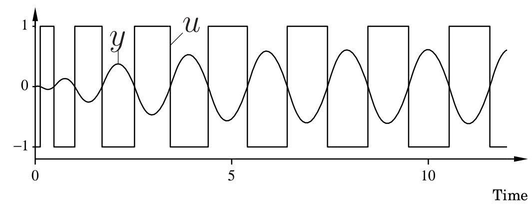
$$G(s) = \frac{(s+10)^2}{(s+1)^3} \quad \text{with feedback} \quad u = -\text{sgn}\, y$$

gives one stable and one unstable limit cycle. The left most intersection corresponds to the stable one.

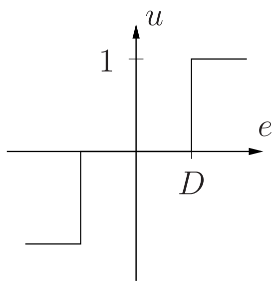
Automatic Tuning of PID Controller

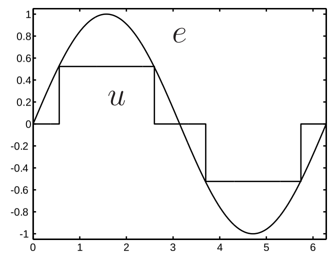
Period and amplitude of relay feedback limit cycle can be used for autotuning:





Describing Function for a Quantizer





Let $e(t) = A\sin\omega t = A\sin\phi$. Then for $\phi \in (0,\pi)$

$$u(\phi) = \begin{cases} 0, & \phi \in (0,\phi_0) \\ 1, & \phi \in (\phi_0, \pi-\phi_0) \end{cases}$$

where $\phi_0 = \arcsin D/A$.

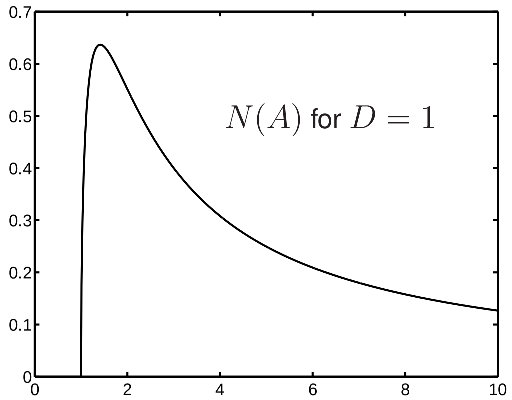
$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi\, d\phi = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi d\phi = \frac{4}{\pi}\int_{\phi_0}^{\pi/2} \sin\phi d\phi$$

$$= \frac{4}{\pi}\cos\phi_0 = \frac{4}{\pi}\sqrt{1 - D^2/A^2}$$

$$N(A) = \begin{cases} 0, & A < D \\ \frac{4}{\pi A}\sqrt{1 - D^2/A^2}, & A \ge D \end{cases}$$

Plot of Describing Function Quantizer



Notice that $N(A) \approx 1.3/A$ for large amplitudes

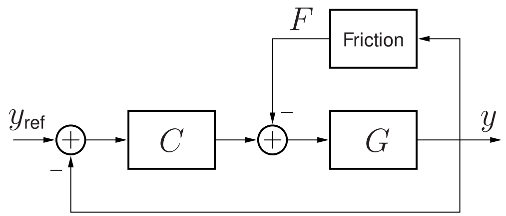
Describing Function Pitfalls

Describing function analysis can give erroneous results.

- A DF may predict a limit cycle even if one does not exist.
- A limit cycle may exist even if the DF does not predict it.
- The predicted amplitude and frequency are only approximations and can be far from the true values.

Accuracy of Describing Function Analysis

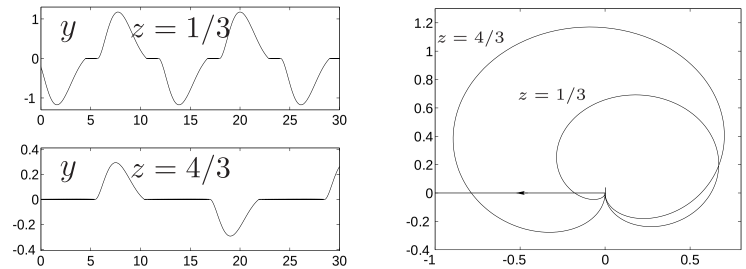
Control loop with friction $F = \text{sgn}\, y$:



Corresponds to

$$\frac{G}{1+GC} = \frac{s(s-z)}{s^3+2s^2+2s+1} \quad \text{with feedback} \quad u = -\text{sgn}\, y$$

The oscillation depends on the zero at $s = z$.

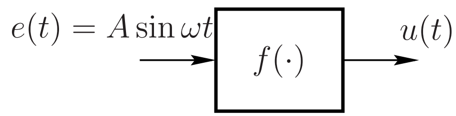


DF gives period times and amplitudes $(T, A) = (11.4, 1.00)$ and $(17.3, 0.23)$, respectively.

Accurate results only if y is close to sinusoidal!

2 minute exercise: *What is $N(A)$ for $f(x) = x^2$?*

Harmonic Balance



A few more Fourier coefficients in the truncation

$$\widehat{u}_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k} (a_n \cos n\omega t + b_n \sin n\omega t)$$

may give much better result. Describing function corresponds to $k = 1$ and $a_0 = 0$.

Example: $f(x) = x^2$ gives $u(t) = (1 - \cos 2\omega t)/2$. Hence by considering $a_0 = 1$ and $a_2 = 1/2$ we get the exact result.

Analysis of Oscillations—A Summary

Time-domain:

- Poincaré maps and Lyapunov functions
- Rigorous results but only for simple examples
- Hard to use for large problems

Frequency-domain:

- Describing function analysis
- Approximate results
- Powerful graphical methods

Today's Goal

You should be able to

- Derive describing functions for static nonlinearities
- Analyze existence and stability of periodic solutions by describing function analysis

EL2620 Nonlinear Control

Lecture 7

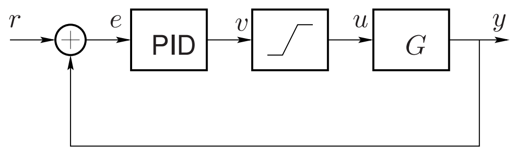
- Compensation for saturation (anti-windup)
- Friction models
- Compensation for friction

Today's Goal

You should be able to analyze and design

- Anti-windup for PID and state-space controllers
- Compensation for friction

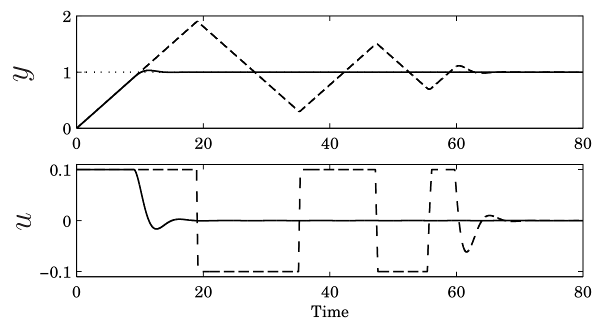
The Problem with Saturating Actuator



- The feedback path is **broken** when u saturates $\Rightarrow$ Open loop behavior!
- Leads to problem when system and/or the *controller* are unstable
  - Example: I-part in PID

Recall: $C_{\text{PID}}(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right)$

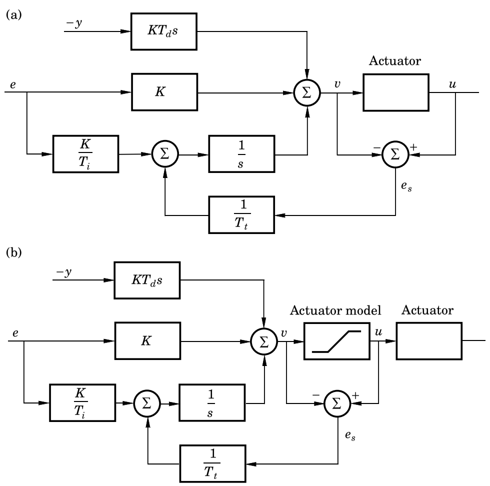
Example—Windup in PID Controller



PID controller without (dashed) and with (solid) anti-windup

Anti-Windup for PID Controller

Anti-windup (a) with actuator output available and (b) without



Anti-Windup is Based on Tracking

When the control signal saturates, the integration state in the controller *tracks* the proper state

The tracking time T_t is the design parameter of the anti-windup

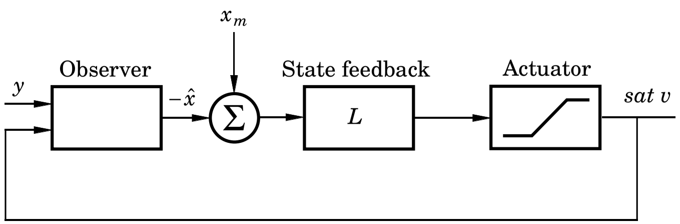
Common choices of T_t:

- $T_t = T_i$
- $T_t = \sqrt{T_i T_d}$

Remark: If $0 < T_t \ll T_i$, then the integrator state becomes sensitive to the instances when $e_s \neq 0$:

$$I(t) = \int_0^t \left[\frac{Ke(\tau)}{T_i} + \frac{e_s(\tau)}{T_t} \right] d\tau \approx \frac{1}{T_t} \int_0^t e_s(\tau)\, d\tau$$

Anti-Windup for Observer-Based State Feedback Controller



$$\dot{\hat{x}} = A\hat{x} + B\,\mathrm{sat}\,v + K(y - C\hat{x})$$
$$v = L(x_m - \hat{x})$$

$\hat{x}$ is estimate of process state, x_m desired (model) state

Need actuator model if $\mathrm{sat}\,v$ is not measurable

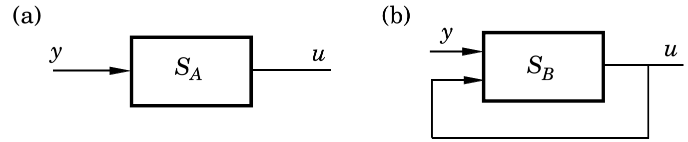
Anti-Windup for General State-Space Controller

State-space controller:

$$\dot{x}_c = Fx_c + Gy$$
$$u = Cx_c + Dy$$

Windup possible if F unstable and u saturates

Idea: Rewrite representation of control law from (a) to (b) with the same input–output relation, but where the unstable S_A is replaced by a stable S_B. If u saturates, then (b) behaves better than (a).



Mimic the observer-based controller:

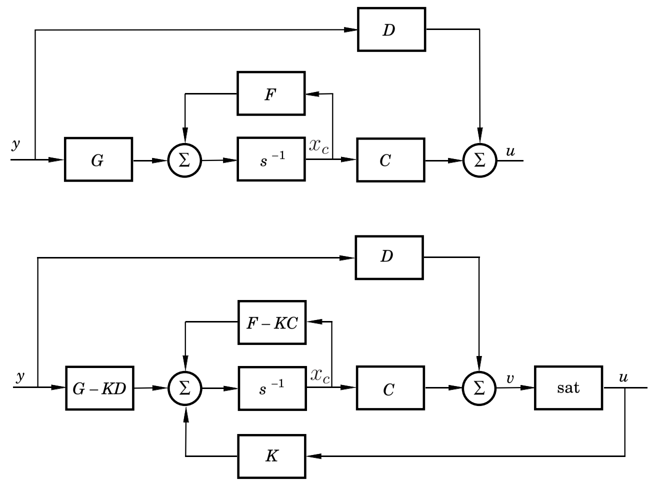
$$\begin{aligned}\dot{x}_c &= Fx_c + Gy + K(u - Cx_c - Dy)\\ &= (F - KC)x_c + (G - KD)y + Ku\end{aligned}$$

Choose K such that $F_0 = F - KC$ has desired (stable) eigenvalues. Then use controller

$$\begin{aligned}\dot{x}_c &= F_0 x_c + G_0 y + Ku\\ u &= \mathrm{sat}(Cx_c + Dy)\end{aligned}$$

where $G_0 = G - KD$.

State-space controller without and with anti-windup:



Controllers with ”Stable” Zeros

Most controllers are minimum phase, i.e. have zeros strictly in LHP

$$\begin{aligned}\dot{x}_c &= Fx_c + Gy \\ u &= Cx_c + Dy\end{aligned} \quad \Rightarrow_{u=0} \quad \begin{aligned}\dot{x}_c &= \overbrace{(F - GC/D)}^{\texttt{zero dynamics}} x_c \\ y &= -C/Dx_c\end{aligned}$$

Thus, choose ”observer” gain

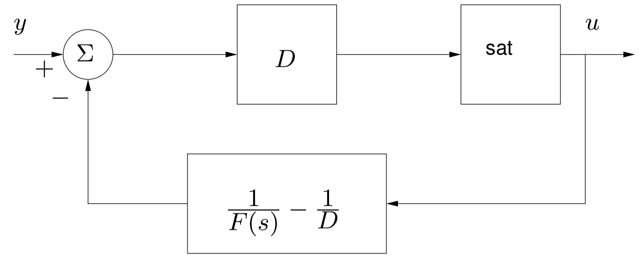
$$K = G/D \quad \Rightarrow \quad F - KC = F - GC/D$$

and the eigenvalues of the ”observer” based controller becomes equal to the zeros of $F(s)$ when u saturates

Note that this implies $G - KD = 0$ in the figure on the previous slide, and we thus obtain P-feedback with gain D under saturation.

Controller $F(s)$ with ”Stable” Zeros

Let $D = \lim_{s\to\infty} F(s)$ and consider the feedback implementation

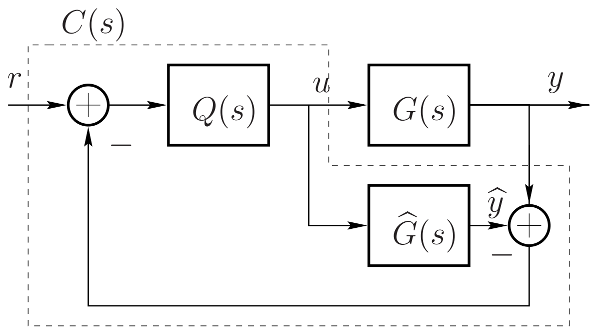


It is easy to show that transfer function from y to u with no saturation equals $F(s)$!

If the transfer function $(1/F(s) - 1/D)$ in the feedback loop is stable (stable zeros) $\Rightarrow$ No stability problems in case of saturation

Internal Model Control (IMC)

IMC: apply feedback only when system G and model $\hat{G}$ differ!



Assume G stable. Note: feedback from the model error $y - \hat{y}$.

Design: assume $\hat{G} \approx G$ and choose Q stable with $Q \approx G^{-1}$.

Example

$$\hat{G}(s) = \frac{1}{T_1 s + 1}$$

Choose

$$Q = \frac{T_1 s + 1}{\tau s + 1}, \quad \tau < T_1$$

Gives the controller

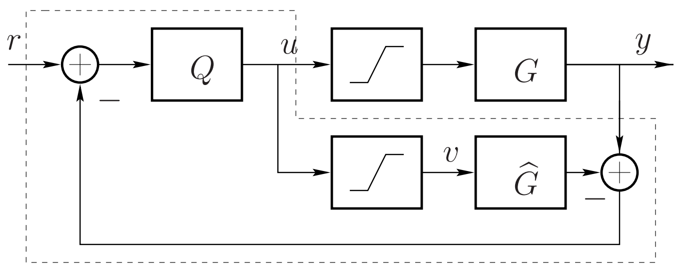
$$F = \frac{Q}{1 - Q\hat{G}} \quad \Rightarrow$$

$$F = \frac{T_1 s + 1}{\tau s} = \frac{T_1}{\tau}\left(1 + \frac{1}{T_1 s}\right)$$

PI-controller!

IMC with Static Nonlinearity

Include nonlinearity in model



Choose $Q \approx G^{-1}$.

Example (cont'd)

Assume $r = 0$ and abuse of Laplace transform notation

$$u = -Q(y - \hat{G}v) = -\frac{T_1 s + 1}{\tau s + 1} y + \frac{1}{\tau s + 1} v$$

if $|u| < u_{\text{max}}$ ($v = u$): PI controller $u = \frac{-(T_1 s + 1)}{\tau s} y$

If $|u| > u_{\text{max}}$ ($v = \pm u_{\text{max}}$):

$$u = -\frac{T_1 s + 1}{\tau s + 1} y \pm \frac{u_{\text{max}}}{\tau s + 1}$$

No integration.

An alternative way to implement anti-windup!

Other Anti-Windup Solutions

Solutions above are all based on tracking.

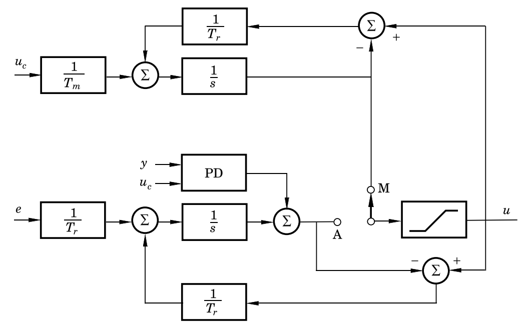
Other solutions include:

- Tune controller to avoid saturation
- Don't update controller states at saturation
- Conditionally reset integration state to zero
- Apply optimal control theory (Lecture 12)

Bumpless Transfer

Another application of the tracking idea is in the switching between automatic and manual control modes.

PID with anti-windup and bumpless transfer:



Note the incremental form of the manual control mode ($\dot{u} \approx u_c/T_m$)

Friction

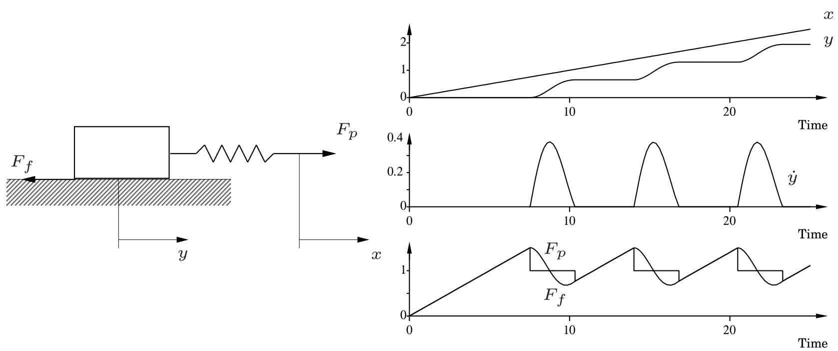
Friction is present almost everywhere

- Often bad:
  - Friction in valves and other actuators
- Sometimes good:
  - Friction in brakes
- Sometimes too small:
  - Earthquakes

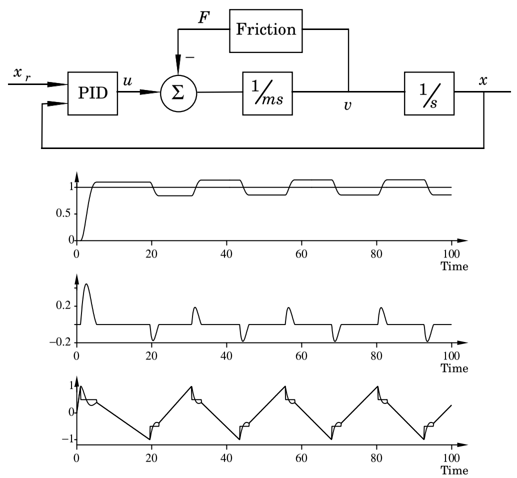
Problems:

- How to model friction?
- How to compensate for friction?
- How to detect friction in control loops?

Stick-Slip Motion

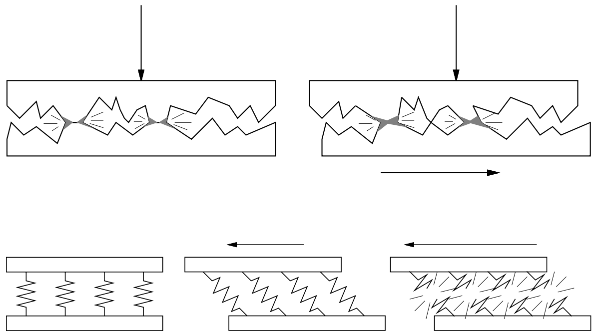


Position Control of Servo with Friction



5 minute exercise: *Which are the signals in the previous plots?*

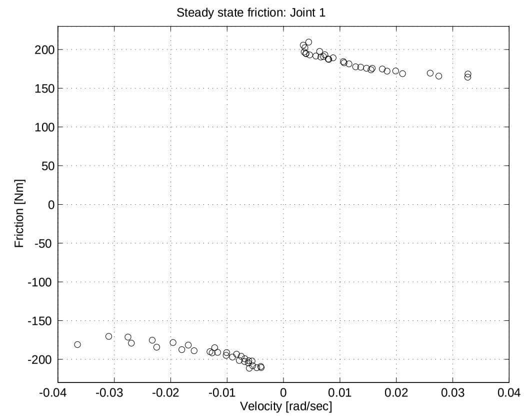
Friction Modeling



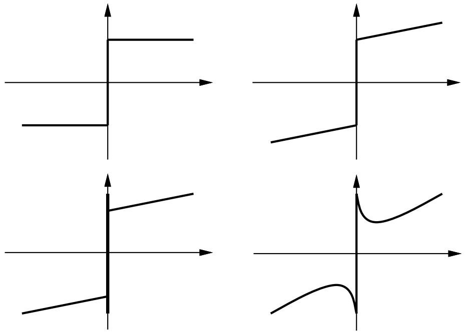
Stribeck Effect

Friction increases with decreasing velocity (for low velocities)

Stribeck (1902)



Classical Friction Models



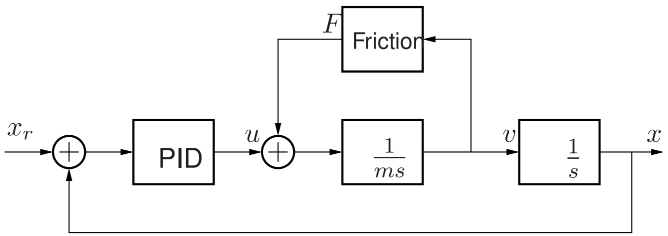
Advanced models capture various friction phenomena better

Friction Compensation

- Lubrication
- Integral action
- Dither signal
- Model-based friction compensation
- Adaptive friction compensation
- The Knocker

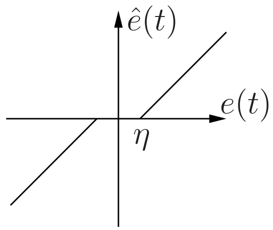
Integral Action

- Integral action compensates for any external disturbance
- Works if friction force changes slowly ($v(t) \approx$ const)
- If friction force changes quickly, then large integral action (small T_i) necessary. May lead to stability problem



Modified Integral Action

Modify the integral part to $I = \frac{K}{T_i} \int^t \hat{e}(\tau) d\tau$ where

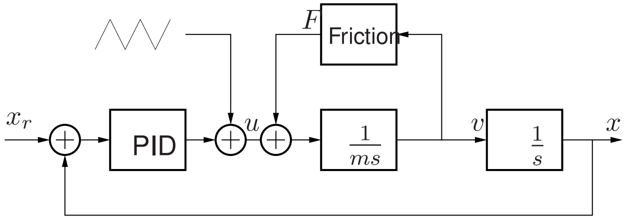


Advantage: Avoid that small static error introduces oscillation

Disadvantage: Error won't go to zero

Dither Signal

Avoid sticking at $v = 0$ (where there is high friction)
by adding high-frequency mechanical vibration (*dither*)



Cf., mechanical maze puzzle
(*labyrintspel*)

Model-Based Friction Compensation

For process with friction F:

$$m\ddot{x} = u - F$$

use control signal

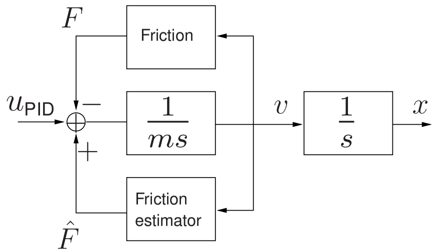
$$u = u_{\text{PID}} + \hat{F}$$

where u_{PID} is the regular control signal and $\hat{F}$ an estimate of F.

Possible if:

- An estimate $\hat{F} \approx F$ is available
- u and F does apply at the same point

Adaptive Friction Compensation



Coulomb friction model: $F = a \operatorname{sgn} v$

Friction estimator:

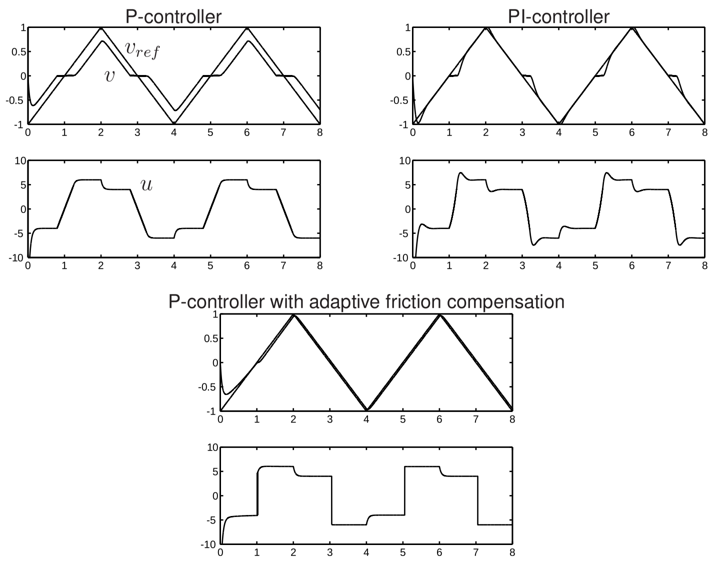
$$\dot{z} = k u_{\text{PID}} \operatorname{sgn} v$$
$$\hat{a} = z - km|v|$$
$$\hat{F} = \hat{a} \operatorname{sgn} v$$

Adaptation converges: $e = a - \hat{a} \to 0$ as $t \to \infty$

Proof:

$$\begin{aligned}\frac{de}{dt} = -\frac{d\hat{a}}{dt} &= -\frac{dz}{dt} + km\frac{d}{dt}|v| \\ &= -k u_{\text{PID}} \operatorname{sgn} v + km\dot{v} \operatorname{sgn} v \\ &= -k \operatorname{sgn} v (u_{\text{PID}} - m\dot{v}) \\ &= -k \operatorname{sgn} v (F - \hat{F}) \\ &= -k(a - \hat{a}) \\ &= -ke\end{aligned}$$

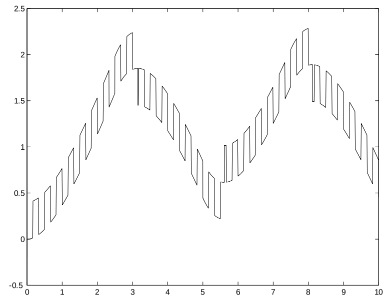
Remark: Careful with $\frac{d}{dt}|v|$ at $v = 0$.



The Knocker

Coulomb friction compensation and square wave dither

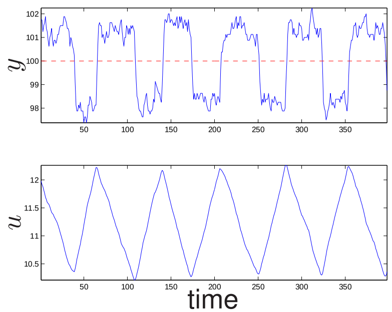
Typical control signal u



Hägglund: Patent and Innovation Cup winner

Detection of Friction in Control Loops

- Friction is due to wear and increases with time
- **Q:** When should valves be maintained?
- **Idea:** Monitor loops automatically and estimate friction



Horch: PhD thesis (2000) and patent

Today's Goal

You should be able to analyze and design

- Anti-windup for PID, state-space, and polynomial controllers
- Compensation for friction

Next Lecture

- Backlash
- Quantization

EL2620 Nonlinear Control

Lecture 8

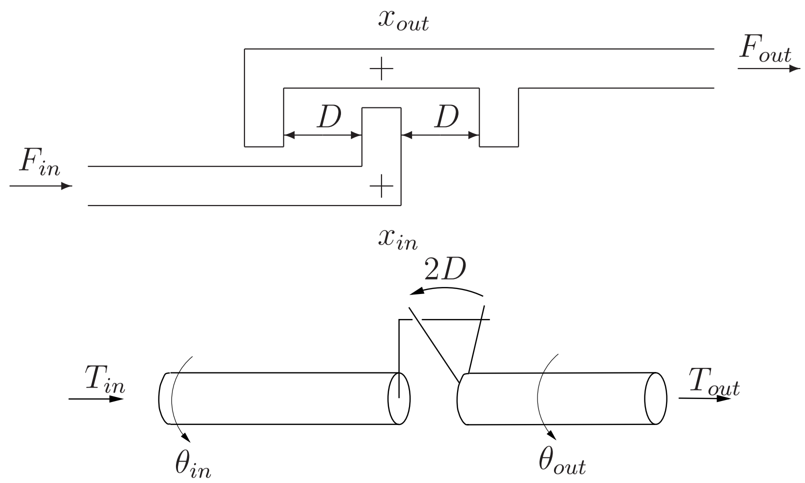
- Backlash
- Quantization

Today's Goal

You should be able to analyze and design for

- Backlash
- Quantization

Linear and Angular Backlash



Backlash

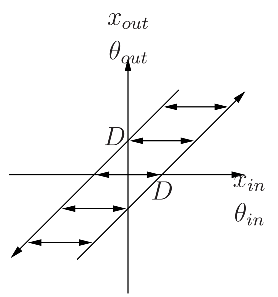
Backlash (*glapp*) is

- present in most mechanical and hydraulic systems
- increasing with wear
- necessary for a gearbox to work in high temperature
- bad for control performance
- sometimes inducing oscillations

Backlash Model

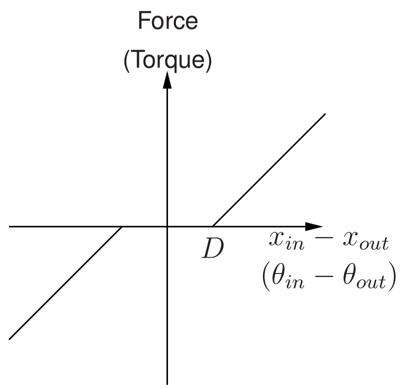
$$\dot{x}_{out} = \begin{cases} \dot{x}_{in}, & \text{in contact} \\ 0, & \text{otherwise} \end{cases}$$

“in contact” if $|x_{out} - x_{in}| = D$ and $\dot{x}_{in}(x_{in} - x_{out}) > 0$.



- Multivalued output; current output depends on history. Thus, backlash is a dynamic phenomena.

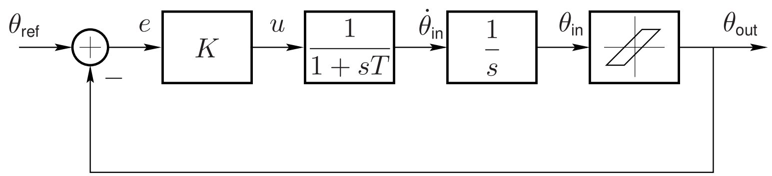
Alternative Model

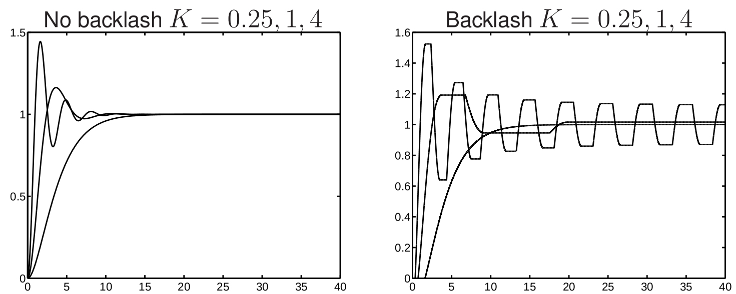


Not equivalent to “Backlash Model”

Effects of Backlash

P-control of motor angle with gearbox having backlash with $D = 0.2$





- Oscillations for $K = 4$ but not for $K = 0.25$ or $K = 1$. Why?
- Note that the amplitude will decrease with decreasing D, but never vanish

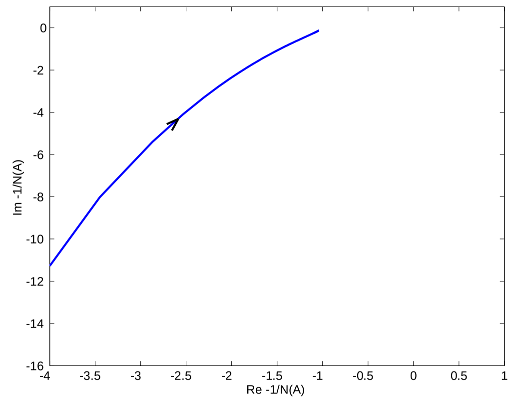
Describing Function for Backlash

If $A < D$ then $N(A) = 0$ else

$$\operatorname{Re} N(A) = \frac{1}{\pi}\left[\frac{\pi}{2} + \arcsin(1 - 2D/A) + 2(1 - 2D/A)\sqrt{\frac{D}{A}\left(1 - \frac{D}{A}\right)}\right]$$

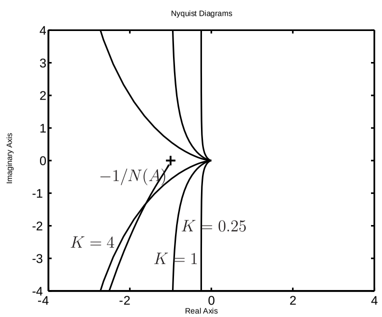
$$\operatorname{Im} N(A) = -\frac{4D}{\pi A}\left(1 - \frac{D}{A}\right)$$

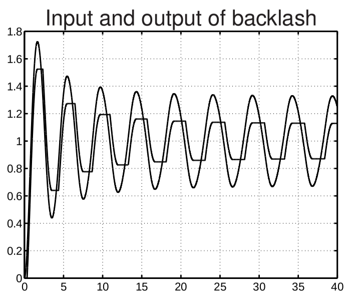
$-1/N(A)$ for $D = 0.2$:



Note that $-1/N(A) \to -1$ as $A \to \infty$ (physical interpretation?)

Describing Function Analysis





$K = 4, D = 0.2$:

DF analysis: Intersection at $A = 0.33, \omega = 1.24$

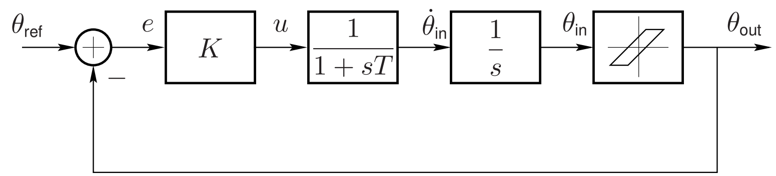
Simulation: $A = 0.33, \omega = 2\pi/5.0 = 1.26$

Describing function predicts oscillation well

Stability Proof for Backlash System

The describing function method is only approximate.

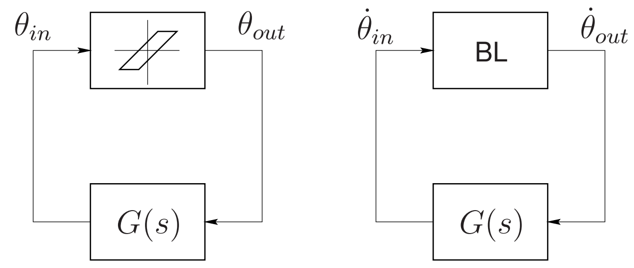
Do there exist conditions that **guarantee** stability (of the steady-state)?



Note that θ_{in} and θ_{out} will not converge to zero

Q: What about $\dot{\theta}_{in}$ and $\dot{\theta}_{out}$?

Rewrite the system:

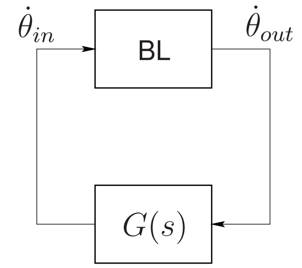


The block “BL” satisfies

$$\dot\theta_{out} = \begin{cases} \dot\theta_{in} & \text{in contact} \\ 0 & \text{otherwise} \end{cases}$$

Homework 2

Analyze this backlash system with input–output stability results:



Passivity Theorem BL is passive

Small Gain Theorem BL has gain $\gamma(BL) = 1$

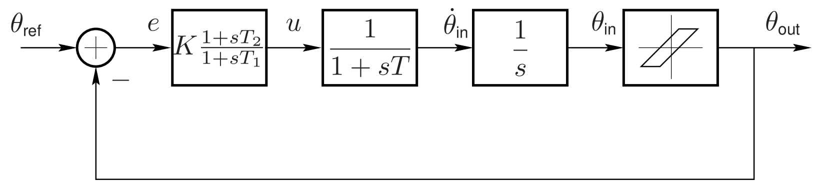
Circle Criterion BL contained in sector $[0, 1]$

Backlash Compensation

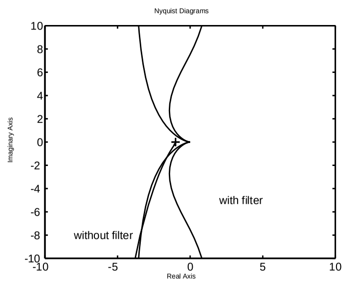
- Mechanical solutions
- Deadzone
- Linear controller design
- Backlash inverse

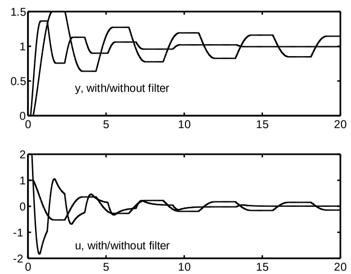
Linear Controller Design

Introduce phase lead compensation:



$F(s) = K\frac{1+sT_2}{1+sT_1}$ with $T_1 = 0.5, T_2 = 2.0$:

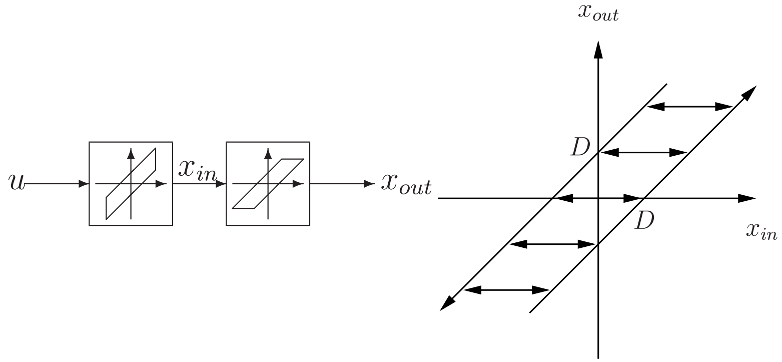


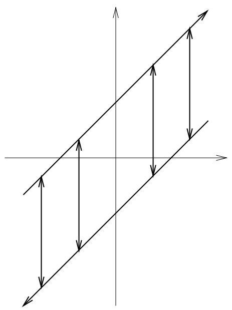


Oscillation removed!

Backlash Inverse

Idea: Let x_{in} jump $\pm 2D$ when $\dot{x}_{out}$ should change sign



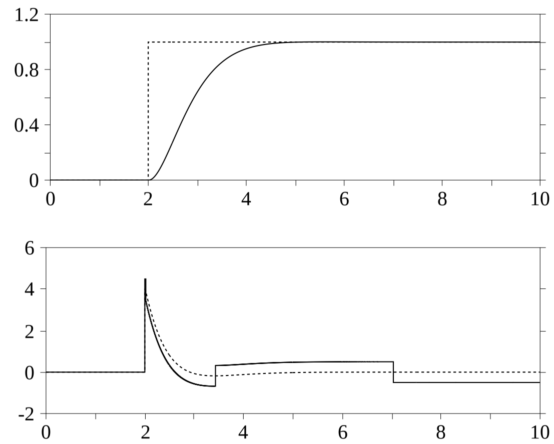


$$x_{in}(t) = \begin{cases} u + \widehat{D} & \text{if } u(t) > u(t-) \\ u - \widehat{D} & \text{if } u(t) < u(t-) \\ x_{in}(t-) & \text{otherwise} \end{cases}$$

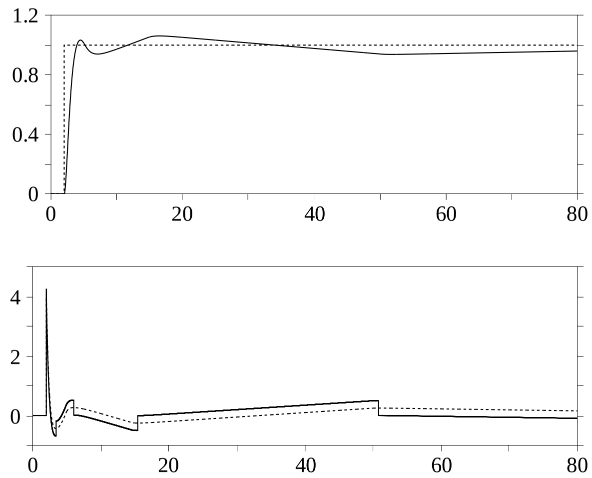
- If $\widehat{D} = D$ then perfect compensation ($x_{out} = u$)
- If $\widehat{D} < D$ then under-compensation (decreased backlash)
- If $\widehat{D} > D$ then over-compensation (may give oscillation)

Example—Perfect Compensation

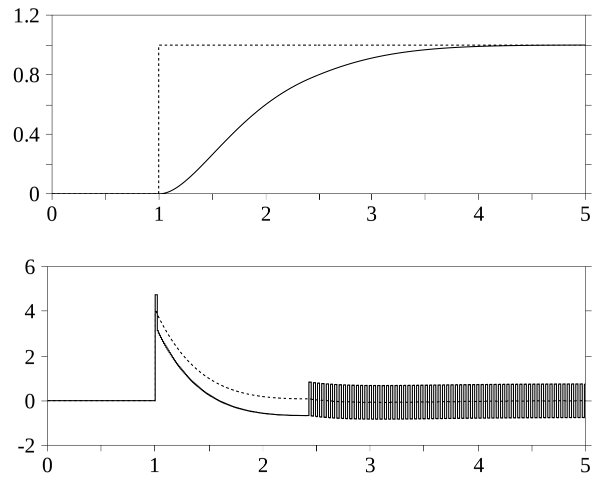
Motor with backlash on input in feedback with PD-controller



Example—Under-Compensation

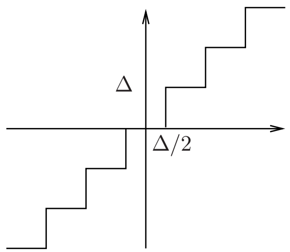


Example—Over-Compensation



Quantization

- What precision is needed in A/D and D/A converters? (8–14 bits?)
- What precision is needed in computations? (8–64 bits?)

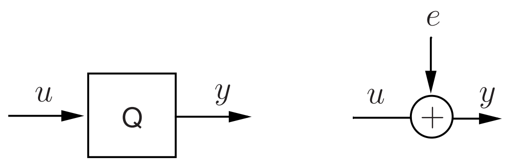


- Quantization in A/D and D/A converters
- Quantization of parameters
- Roundoff, overflow, underflow in computations

Linear Model of Quantization

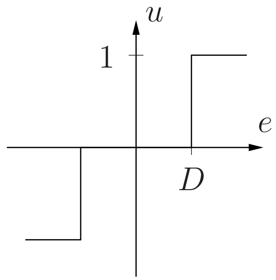
Model quantization error as a uniformly distributed stochastic signal e independent of u with

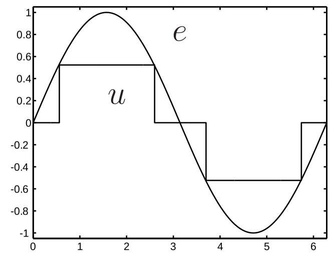
$$\mathrm{Var}(e) = \int_{-\infty}^{\infty} e^2 f_e \, de = \int_{-\Delta/2}^{\Delta/2} \frac{e^2}{\Delta} \, de = \frac{\Delta^2}{12}$$



- May be reasonable if Δ is small compared to the variations in u
- But, added noise can never affect stability while quantization can!

Describing Function for Deadzone Relay

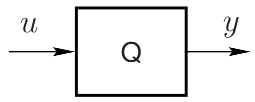


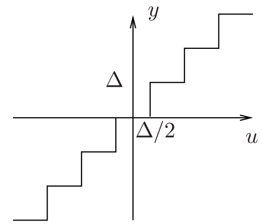


Lecture 6 $\Rightarrow$

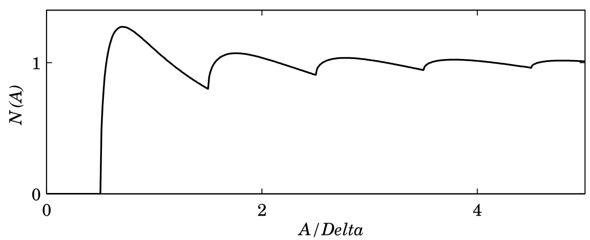
$$N(A) = \begin{cases} 0, & A < D \\ \dfrac{4}{\pi A}\sqrt{1 - D^2/A^2}, & A > D \end{cases}$$

Describing Function for Quantizer





$$N(A) = \begin{cases} 0, & A < \frac{\Delta}{2} \\ \dfrac{4\Delta}{\pi A}\sum_{i=1}^{n}\sqrt{1 - \left(\dfrac{2i-1}{2A}\Delta\right)^2}, & \frac{2n-1}{2}\Delta < A < \frac{2n+1}{2}\Delta \end{cases}$$



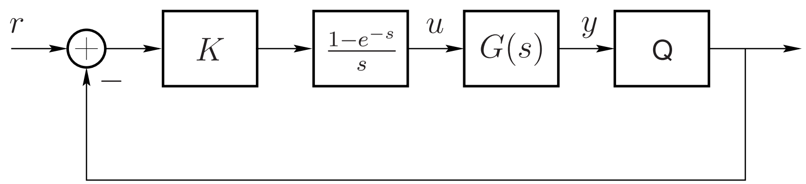
- The maximum value is $4/\pi \approx 1.27$ attained at $A \approx 0.71\Delta$.
- Predicts oscillation if Nyquist curve intersects negative real axis to the left of $-\pi/4 \approx -0.79$
- Controller with gain margin $> 1/0.79 = 1.27$ avoids oscillation
- Reducing Δ reduces only the oscillation amplitude

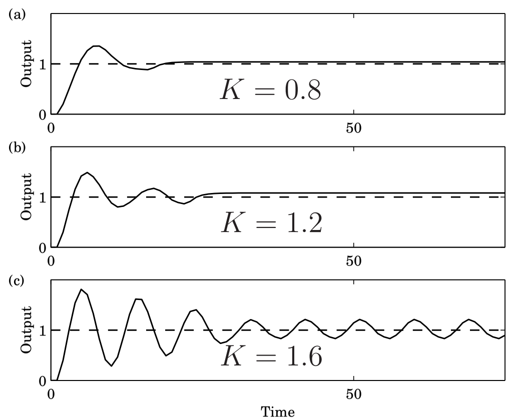
Example—Motor with P-controller.

Quantization of process output with $\Delta = 0.2$

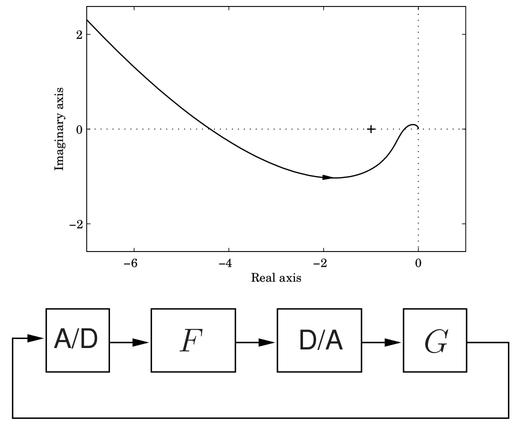
Nyquist of linear part (K & ZOH & $G(s)$) intersects at $-0.5K$: Stability for $K < 2$ without Q.

Stable oscillation predicted for $K > 2/1.27 = 1.57$ with Q.

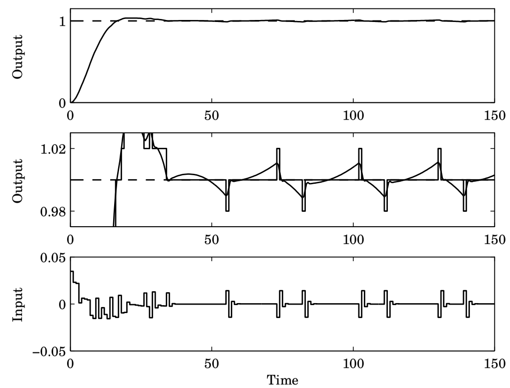




Example—$1/s^2$ & 2nd-Order Controller



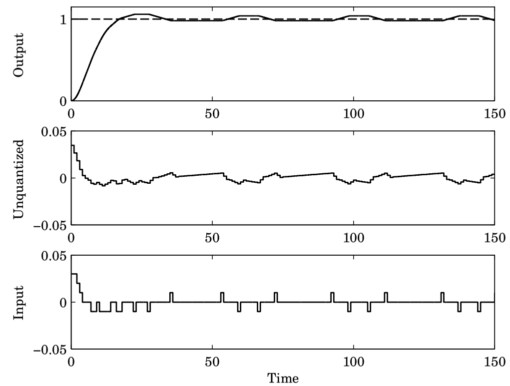
Quantization $\Delta = 0.02$ in A/D converter:



Describing function: $A_y = 0.01$ and $T = 39$

Simulation: $A_y = 0.01$ and $T = 28$

Quantization $\Delta = 0.01$ in D/A converter:

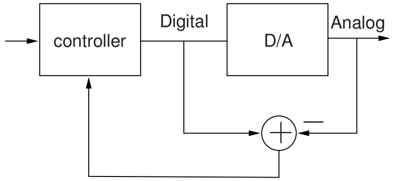


Describing function: $A_u = 0.005$ and $T = 39$

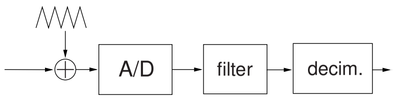
Simulation: $A_u = 0.005$ and $T = 39$

Quantization Compensation

- Improve accuracy (larger word length)
- Avoid unstable controller and gain margins < 1.3
- Use the tracking idea from anti-windup to improve D/A converter



- Use analog dither, oversampling, and digital lowpass filter to improve accuracy of A/D converter



Today’s Goal

You should now be able to analyze and design for

- Backlash
- Quantization

EL2620 Nonlinear Control

Lecture 9

- Nonlinear control design based on high-gain control

Today's Goal

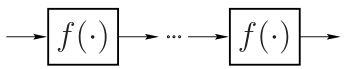
You should be able to analyze and design

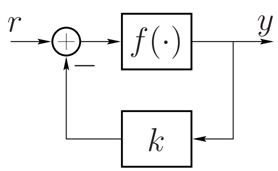
- High-gain control systems
- Sliding mode controllers

History of the Feedback Amplifier

New York–San Francisco communication link 1914.

High signal amplification with low distortion was needed.

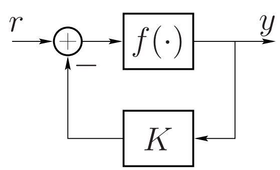


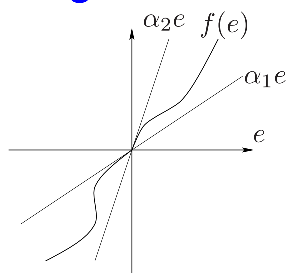


Feedback amplifiers were the solution!

Black, Bode, and Nyquist at Bell Labs 1920–1950.

Linearization Through High Gain Feedback





$$\alpha_1 \le \frac{f(e)}{e} \le \alpha_2 \quad \Rightarrow \quad \frac{\alpha_1}{1+\alpha_1 K} r \le y \le \frac{\alpha_2}{1+\alpha_2 K} r$$

choose $K \gg 1/\alpha_1$, yields

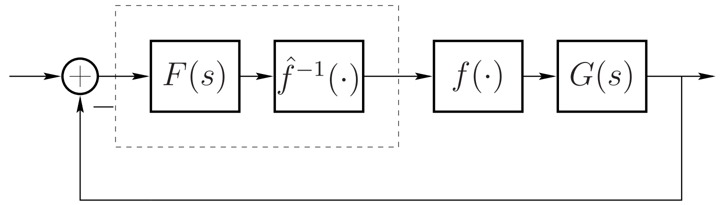
$$y \approx \frac{1}{K} r$$

A Word of Caution

Nyquist: high loop-gain may induce oscillations (due to dynamics)!

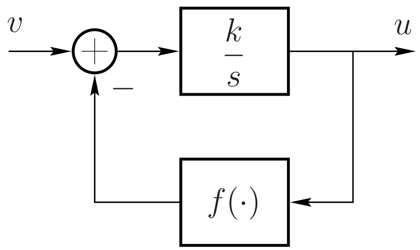
Inverting Nonlinearities

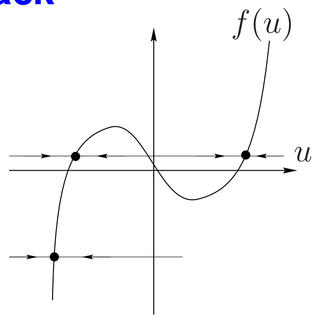
Compensation of static nonlinearity through inversion:



Should be combined with feedback as in the figure!

Remark: How to Obtain f^{-1} from f using Feedback



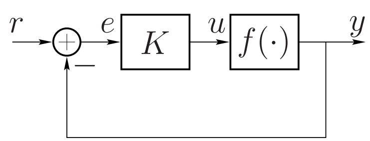


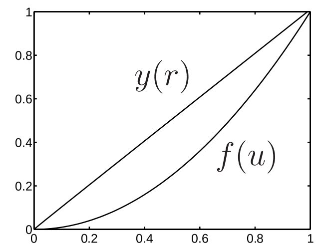
$$\dot{u} = k\big(v - f(u)\big)$$

If $k > 0$ large and $df/du > 0$, then $\dot{u} \to 0$ and

$$0 = k\big(v - f(u)\big) \quad \Leftrightarrow \quad f(u) = v \quad \Leftrightarrow \quad u = f^{-1}(v)$$

Example—Linearization of Static Nonlinearity

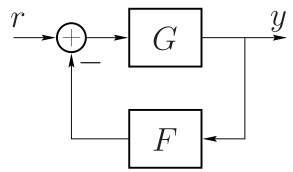




Linearization of $f(u) = u^2$ through feedback.

The case $K = 100$ is shown in the plot: $y(r) \approx r$.

The Sensitivity Function $S = (1+GF)^{-1}$



The closed-loop system is

$$G_{cl} = \frac{G}{1+GF}$$

Small perturbations dG in G gives

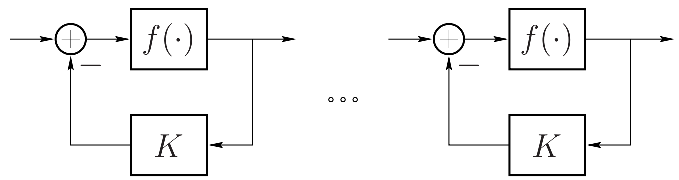
$$\frac{dG_{cl}}{dG} = \frac{1}{(1+GF)^2} \quad \Rightarrow \quad \frac{dG_{cl}}{G_{cl}} = \frac{1}{1+GF}\frac{dG}{G} = S\frac{dG}{G}$$

S is the closed-loop **sensitivity** to open-loop perturbations.

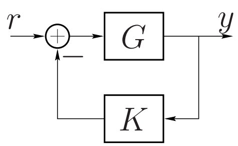
Distortion Reduction via Feedback

The feedback reduces distortion in each link.

Several links give distortion-free high gain.



Example—Distortion Reduction



Let $G = 1000$,
distortion $dG/G = 0.1$

Choose $K = 0.1 \quad \Rightarrow \quad S = (1+GK)^{-1} \approx 0.01$. Then

$$\frac{dG_{cl}}{G_{cl}} = S\frac{dG}{G} \approx 0.001$$

100 feedback amplifiers in series give total amplification

$$G_{tot} = (G_{cl})^{100} \approx 10^{100}$$

and total distortion

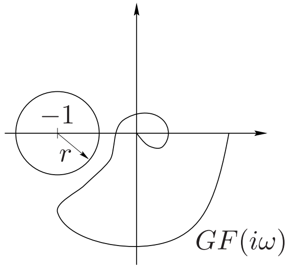
$$\frac{dG_{tot}}{G_{tot}} = (1+10^{-3})^{100} - 1 \approx 0.1$$

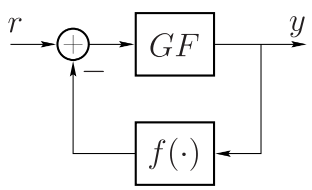
Transcontinental Communication Revolution

The feedback amplifier was patented by Black 1937.

Year	*Channels*	*Loss (dB)*	*No amp's*
1914	1	60	3–6
1923	1–4	150–400	6–20
1938	16	1000	40
1941	480	30000	600

Sensitivity and the Circle Criterion





Consider a circle $\mathcal{C} := \{z \in \mathbb{C} : |z+1| = r\}, r \in (0,1)$.

$GF(i\omega)$ stays outside $\mathcal{C}$ if

$$|1 + GF(i\omega)| > r \qquad \Leftrightarrow \qquad |S(i\omega)| \le r^{-1}$$

Then, the Circle Criterion gives stability if $\dfrac{1}{1+r} \le \dfrac{f(y)}{y} \le \dfrac{1}{1-r}$

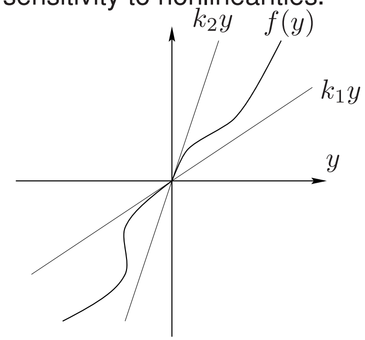
Small Sensitivity Allows Large Uncertainty

If $|S(i\omega)|$ is small, we can choose r large (close to one).

This corresponds to a large sector for $f(\cdot)$.

Hence, $|S(i\omega)|$ small implies low sensitivity to nonlinearities.

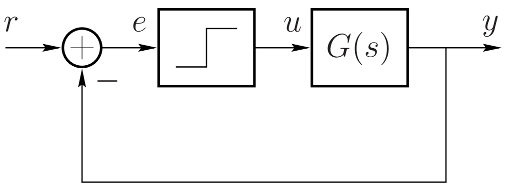
$$k_1 = \frac{1}{1+r}$$
$$k_2 = \frac{1}{1-r}$$



On–Off Control

On–off control is the simplest control strategy.

Common in temperature control, level control etc.



The relay corresponds to infinitely high gain at the switching point.

A Control Design Idea

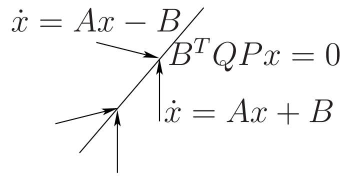
Assume $V(x) = x^T P x$, $P = P^T > 0$, represents the energy of

$$\dot{x} = Ax + Bu, \qquad u \in [-1, 1]$$

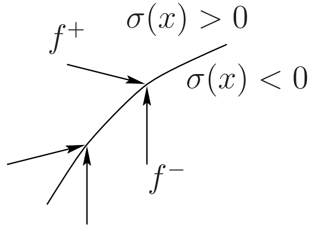
Choose u such that V decays as fast as possible:

$$\dot{V} = x^T(A^T P + PA)x + 2B^T P x u$$

is minimized if $u = -\operatorname{sgn}(B^T P x)$ (Notice that $\dot{V} = a + bu$, i.e. just a segment of line in u, $-1 < u < 1$. Hence the lowest value is at an endpoint, depending on the sign of the slope b.)

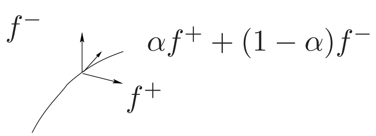


Sliding Modes



$$\dot{x} = \begin{cases} f^+(x), & \sigma(x) > 0 \\ f^-(x), & \sigma(x) < 0 \end{cases}$$

The **sliding mode** is $\dot{x} = \alpha f^+ + (1-\alpha) f^-$, where α satisfies $\alpha f_n^+ + (1-\alpha) f_n^- = 0$ for the normal projections of f^+, f^-



The **sliding surface** is $S = \{x : \sigma(x) = 0\}$.

Example

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u = Ax + Bu$$

$$u = -\operatorname{sgn} \sigma(x) = -\operatorname{sgn} x_2 = -\operatorname{sgn}(Cx)$$

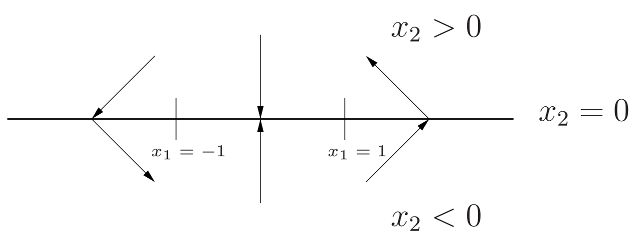
is equivalent to

$$\dot{x} = \begin{cases} Ax - B, & x_2 > 0 \\ Ax + B, & x_2 < 0 \end{cases}$$

For small x_2 we have

$$\begin{cases} \dot{x}_2(t) \approx x_1 - 1, \; \dfrac{dx_2}{dx_1} \approx 1 - x_1 & x_2 > 0 \\ \dot{x}_2(t) \approx x_1 + 1, \; \dfrac{dx_2}{dx_1} \approx 1 + x_1 & x_2 < 0 \end{cases}$$

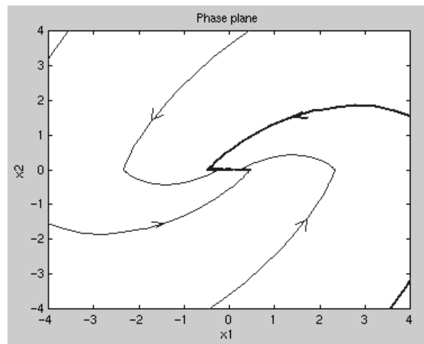
This implies the following behavior



Sliding Mode Dynamics

The dynamics along the sliding surface S is obtained by setting $u = u_{eq} \in [-1, 1]$ such that $x(t)$ stays on S.

u_{eq} is called the **equivalent control**.



Example (cont'd)

Finding $u = u_{\text{eq}}$ such that $\dot{\sigma}(x) = \dot{x}_2 = 0$ on $\sigma(x) = x_2 = 0$ gives

$$0 = \dot{x}_2 = x_1 - \underbrace{x_2}_{=0} + u_{\text{eq}} = x_1 + u_{\text{eq}} \quad \Rightarrow \quad u_{\text{eq}} = -x_1$$

Insert this in the equation for $\dot{x}_1$:

$$\dot{x}_1 = -\underbrace{x_2}_{=0} + u_{\text{eq}} = -x_1$$

gives the dynamics on the sliding surface $S = \{x : \, x_2 = 0\}$.

Deriving the Equivalent Control

Assume

$$\dot{x} = f(x) + g(x)u$$
$$u = -\operatorname{sgn} \sigma(x)$$

has a stable sliding surface $S = \{x : \, \sigma(x) = 0\}$. Then, for $x \in S$,

$$0 = \dot{\sigma}(x) = \frac{d\sigma}{dx} \cdot \frac{dx}{dt} = \frac{d\sigma}{dx}\Big(f(x) + g(x)u\Big)$$

The equivalent control is thus given by

$$u_{\text{eq}} = -\left(\frac{d\sigma}{dx} g(x)\right)^{-1} \frac{d\sigma}{dx} f(x)$$

if the inverse exists.

Equivalent Control for Linear System

$$\dot{x} = Ax + Bu$$
$$u = -\operatorname{sgn} \sigma(x) = -\operatorname{sgn}(Cx)$$

Assume $CB > 0$. The sliding surface $S = \{x : \, Cx = 0\}$ so

$$0 = \dot{\sigma}(x) = \frac{d\sigma}{dx}\Big(f(x) + g(x)u\Big) = C\big(Ax + Bu_{\text{eq}}\big)$$

gives $u_{\text{eq}} = -CAx/CB$.

Example (cont'd): For the example:

$$u_{\text{eq}} = -CAx/CB = -\begin{pmatrix} 1 & -1 \end{pmatrix} x = -x_1,$$

because $\sigma(x) = x_2 = 0$. (Same result as before.)

Sliding Dynamics

The dynamics on $S = \{x : \, Cx = 0\}$ is given by

$$\dot{x} = Ax + Bu_{\text{eq}} = \left(I - \frac{1}{CB} BC\right) Ax,$$

under the constraint $Cx = 0$, where the eigenvalues of $(I - BC/CB)A$ are equal to the zeros of $sG(s) = sC(sI - A)^{-1}B$.

Remark: The condition that $Cx = 0$ corresponds to the zero at $s = 0$, and thus this dynamic disappears on $S = \{x : \, Cx = 0\}$.

Proof

$$\dot{x} = Ax + Bu$$

$$y = Cx \;\Rightarrow\; \dot{y} = CAx + CBu \Rightarrow\; u = \frac{1}{CB}CAx - \frac{1}{CB}\dot{y} \Rightarrow$$

$$\dot{x} = \Big(I - \frac{1}{CB}BC\Big)Ax - \frac{1}{CB}B\dot{y}$$

Hence, the transfer function from $\dot{y}$ to u equals

$$\frac{-1}{CB} + \frac{1}{CB}CA(sI - ((I - \frac{1}{CB}BC)A))^{-1}\frac{-1}{CB}B$$

but this transfer function is also $1/(sG(s))$ Hence, the eigenvalues of $(I - BC/CB)A$ are equal to the zeros of $sG(s)$.

Design of Sliding Mode Controller

Idea: Design a control law that forces the state to $\sigma(x) = 0$. Choose $\sigma(x)$ such that the sliding mode tends to the origin. Assume

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x) + g_1(x)u \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = f(x) + g(x)u$$

Choose control law

$$u = -\frac{p^T f(x)}{p^T g(x)} - \frac{\mu}{p^T g(x)} \operatorname{sgn} \sigma(x),$$

where $\mu > 0$ is a design parameter, $\sigma(x) = p^T x$, and $p^T = \begin{pmatrix} p_1 & \dots & p_n \end{pmatrix}$ are the coefficients of a stable polynomial.

Closed-Loop Stability

Consider $V(x) = \sigma^2(x)/2$ with $\sigma(x) = p^T x$. Then,

$$\dot{V} = \sigma^T(x)\dot{\sigma}(x) = x^T p\big(p^T f(x) + p^T g(x)u\big)$$

With the chosen control law, we get

$$\dot{V} = -\mu\sigma(x) \operatorname{sgn} \sigma(x) < 0$$

so x tend to $\sigma(x) = 0$.

$$\begin{aligned} 0 = \sigma(x) &= p_1 x_1 + \cdots + p_{n-1}x_{n-1} + p_n x_n \\ &= p_1 x_n^{(n-1)} + \cdots + p_{n-1}x_n^{(1)} + p_n x_n^{(0)} \end{aligned}$$

where $x^{(k)}$ denote time derivative. Now p corresponds to a stable differential equation, and $x_n \to 0$ exponentially as $t \to \infty$. The state relations $x_{k-1} = \dot{x}_k$ now give $x \to 0$ exponentially as $t \to \infty$.

Time to Switch

Consider an initial point x_0 such that $\sigma_0 = \sigma(x_0) > 0$. Since

$$\sigma(x)\dot{\sigma}(x) = -\mu\sigma(x) \operatorname{sgn} \sigma(x)$$

it follows that as long as $\sigma(x) > 0$:

$$\dot{\sigma}(x) = -\mu$$

Hence, the time to the first switch ($\sigma(x) = 0$) is

$$t_s = \frac{\sigma_0}{\mu} < \infty$$

Note that $t_s \to 0$ as $\mu \to \infty$.

Example—Sliding Mode Controller

Design state-feedback controller for

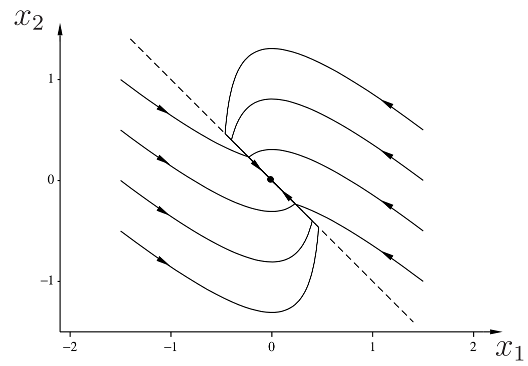
$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

Choose $p_1 s + p_2 = s + 1$ so that $\sigma(x) = x_1 + x_2$. The controller is given by

$$u = -\frac{p^T A x}{p^T B} - \frac{\mu}{p^T B} \operatorname{sgn} \sigma(x)$$
$$= 2x_1 - \mu \operatorname{sgn}(x_1 + x_2)$$

Phase Portrait

Simulation with $\mu = 0.5$. Note the sliding surface $\sigma(x) = x_1 + x_2$.



Time Plots

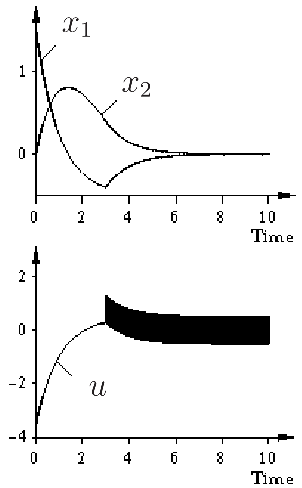
Initial condition $x(0) = \begin{pmatrix} 1.5 & 0 \end{pmatrix}^T$.

Simulation agrees well with time to switch

$$t_s = \frac{\sigma_0}{\mu} = 3$$

and sliding dynamics

$$\dot{y} = -y$$



The Sliding Mode Controller is Robust

Assume that only a model $\dot{x} = \widehat{f}(x) + \widehat{g}(x)u$ of the true system $\dot{x} = f(x) + g(x)u$ is known. Still, however,

$$\dot{V} = \sigma(x) \left[\frac{p^T (f \widehat{g}^T - \widehat{f} g^T) p}{p^T \widehat{g}} - \mu \frac{p^T g}{p^T \widehat{g}} \operatorname{sgn} \sigma(x) \right] < 0$$

if $\operatorname{sgn}(p^T g) = \operatorname{sgn}(p^T \widehat{g})$ and $\mu > 0$ is sufficiently large.

The closed-loop system is thus robust against model errors!

(High gain control with stable open loop zeros)

Comments on Sliding Mode Control

- Efficient handling of model uncertainties
- Often impossible to implement infinite fast switching
- Smooth version through low pass filter or boundary layer
- Applications in robotics and vehicle control
- Compare puls-width modulated control signals

Today's Goal

You should be able to analyze and design

- High-gain control systems
- Sliding mode controllers

Next Lecture

- Lyapunov design methods
- Exact feedback linearization

EL2620 Nonlinear Control

Lecture 10

- Exact feedback linearization
- Input-output linearization
- Lyapunov-based control design methods

Output Feedback and State Feedback

$$\dot{x} = f(x,u)$$
$$y = h(x)$$

- **Output feedback:** Find $u = k(y)$ such that the closed-loop system
$$\dot{x} = f\big(x, k\big(h(x)\big)\big)$$
has nice properties.
- **State feedback:** Find $u = \ell(x)$ such that
$$\dot{x} = f(x, \ell(x))$$
has nice properties.

k and ℓ may include dynamics.

Nonlinear Output Feedback Controllers

- Nonlinear dynamical controller: $\dot{z} = a(z,y)$, $u = c(z)$
- Linear dynamics, static nonlinearity: $\dot{z} = Az + By$, $u = c(z)$
- Linear controller: $\dot{z} = Az + By$, $u = Cz$

Nonlinear Observers

What if x is not measurable?

$$\dot{x} = f(x,u), \quad y = h(x)$$

Simplest observer

$$\dot{\widehat{x}} = f(\widehat{x}, u)$$

Feedback correction, as in linear case,

$$\dot{\widehat{x}} = f(\widehat{x}, u) + K(y - h(\widehat{x}))$$

Choices of K

- Linearize f at x_0, find K for the linearization
- Linearize f at $\widehat{x}(t)$, find $K = K(\widehat{x})$ for the linearization

Second case is called *Extended Kalman Filter*

Some state feedback control approaches

- Exact feedback linearization
- Input-output linearization
- Lyapunov-based design - *backstepping control*

Exact Feedback Linearization

Consider the nonlinear *control-affine* system

$$\dot{x} = f(x) + g(x)u$$

idea: use a state-feedback controller $u(x)$ to make the system linear

Example 1:

$$\dot{x} = \cos x \;-\; x^3 \;+\; u$$

The state-feedback controller

$$u(x) = -\cos x \;+\; x^3 \;-\; kx \;+\; v$$

yields the linear system

$$\dot{x} = -kx \;+\; v$$

Another Example

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + u$$

How do we cancel the term $\sin x_2$?

Perform transformation of states into linearizable form:

$$z_1 = x_1, \quad z_2 = \dot{x}_1 = a \sin x_2$$

yields

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = a \cos x_2(-x_1^2 + u)$$

and the linearizing control becomes

$$u(x) = x_1^2 + \frac{v}{a \cos x_2}, \quad x_2 \in [-\pi/2, \pi/2]$$

Diffeomorphisms

A nonlinear state transformation $z = T(x)$ with

- T invertible for x in the domain of interest
- T and T^{-1} continuously differentiable

is called a *diffeomorphism*

Definition: A nonlinear system

$$\dot{x} = f(x) + g(x)u$$

is **feedback linearizable** if there exist a diffeomorphism T whose domain contains the origin and transforms the system into the form

$$\dot{x} = Ax + B\gamma(x)\,(u - \alpha(x))$$

with (A, B) controllable and $\gamma(x)$ nonsingular for all x in the domain of interest.

Exact vs. Input-Output Linearization

The example again, but now with an output

$$\dot{x}_1 = a\sin x_2, \quad \dot{x}_2 = -x_1^2 + u, \quad y = x_2$$

- The control law $u = x_1^2 + v/a\cos x_2$ yields

  $$\dot{z}_1 = z_2\,; \quad \dot{z}_2 = v\,; \quad y = \sin^{-1}(z_2/a)$$

  which is nonlinear in the output.
- If we want a linear input-output relationship we could instead use

  $$u = x_1^2 + v$$

  to obtain

  $$\dot{x}_1 = a\sin x_2, \quad \dot{x}_2 = v, \quad y = x_2$$

  which is linear from v to y

Caution: the control has rendered the state x_1 *unobservable*

Input-Output Linearization

Use state feedback $u(x)$ to make the control-affine system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}$$

linear from the input v to the output y

The general idea: differentiate the output, $y = h(x)$, p times untill the control u appears explicitly in $y^{(p)}$, and then determine u so that

$$y^{(p)} = v$$

i.e., $G(s) = 1/s^p$

Example: controlled van der Pol equation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2 + u \\ y &= x_1\end{aligned}$$

Differentiate the output

$$\begin{aligned}\dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u\end{aligned}$$

The state feedback controller

$$u = x_1 - \epsilon(1 - x_1^2)x_2 + v \quad \Rightarrow \quad \ddot{y} = v$$

Lie Derivatives

Consider the nonlinear SISO system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u\,; \; x \in \mathbb{R}^n, u \in \mathbb{R}^1 \\ y &= h(x), \quad y \in \mathbb{R}\end{aligned}$$

The derivative of the output

$$\dot{y} = \frac{dh}{dx}\dot{x} = \frac{dh}{dx}\left(f(x) + g(x)u\right) \triangleq L_f h(x) + L_g h(x) u$$

where $L_f h(x)$ and $L_g h(x)$ are *Lie derivatives* ($L_f h$ is the derivative of h along the vector field of $\dot{x} = f(x)$)

Repeated derivatives

$$L_f^k h(x) = \frac{d(L_f^{k-1}h)}{dx} f(x), \quad L_g L_f h(x) = \frac{d(L_f h)}{dx} g(x)$$

Lie derivatives and relative degree

- The relative degree p of a system is defined as the number of integrators between the input and the output (the number of times y must be differentiated for the input u to appear)
- A linear system

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

has relative degree $p = n - m$

- A nonlinear system has relative degree p if

$$L_g L_f^{i-1} h(x) = 0, i = 1, \ldots, p-1\,; \quad L_g L_f^{p-1} h(x) \neq 0 \quad \forall x \in D$$

Example

The controlled van der Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2 + u \\ y &= x_1 \end{aligned}$$

Differentiating the output

$$\begin{aligned} \dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u \end{aligned}$$

Thus, the system has relative degree $p = 2$

The input-output linearizing control

Consider a nth order SISO system with relative degree p

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

Differentiating the output repeatedly

$$\begin{aligned} \dot{y} &= \frac{dh}{dx}\dot{x} = L_f h(x) + \overbrace{L_g h(x) u}^{=0} \\ &\vdots \\ y^{(p)} &= L_f^p h(x) + L_g L_f^{p-1} h(x) u \end{aligned}$$

and hence the state-feedback controller

$$u = \frac{1}{L_g L_f^{p-1} h(x)} \left(-L_f^p h(x) + v\right)$$

results in the linear input-output system

$$y^{(p)} = v$$

Zero Dynamics

- Note that the order of the linearized system is p, corresponding to the relative degree of the system
- Thus, if $p < n$ then $n - p$ states are unobservable in y.
- The dynamics of the $n - p$ states not observable in the linearized dynamics of y are called the **zero dynamics**. Corresponds to the dynamics of the system when y is forced to be zero for all times.
- A system with **unstable zero dynamics** is called non-minimum phase (and should not be input-output linearized!)

van der Pol again

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u$$

- With $y = x_1$ the relative degree $p = n = 2$ and there are no zero dynamics, thus we can transform the system into $\ddot{y} = v$.
- With $y = x_2$ the relative degree $p = 1 < n$ and the zero dynamics are given by $\dot{x}_1 = 0$, which is not asymptotically stable (but bounded)

Lyapunov-Based Control Design Methods

$$\dot{x} = f(x, u)$$

- Find stabilizing state feedback $u = u(x)$
- Verify stability through Control Lyapunov function
- Methods depend on structure of f

Here we limit discussion to **Back-stepping control design**, which require certain f discussed later.

A simple introductory example

Consider

$$\dot{x} = \cos x - x^3 + u$$

Apply the linearizing control

$$u = -\cos x + x^3 - kx$$

Choose the Lyapunov candidate $V(x) = x^2/2$

$$V(x) > 0\,, \quad \dot{V} = -kx^2 < 0$$

Thus, the system is globally asymptotically stable

But, the term x^3 in the control law may require large control moves!

The same example

$$\dot{x} = \cos x - x^3 + u$$

Now try the control law

$$u = -\cos x - kx$$

Choose the same Lyapunov candidate $V(x) = x^2/2$

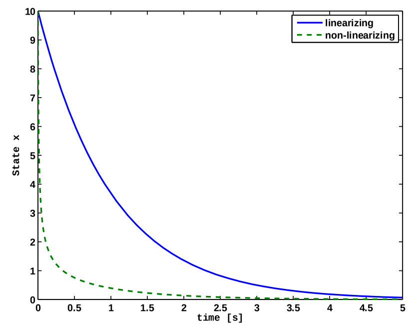
$$V(x) > 0\,, \quad \dot{V} = -x^4 - kx^2 < 0$$

Thus, also globally asymptotically stable (and more negative $\dot{V}$)

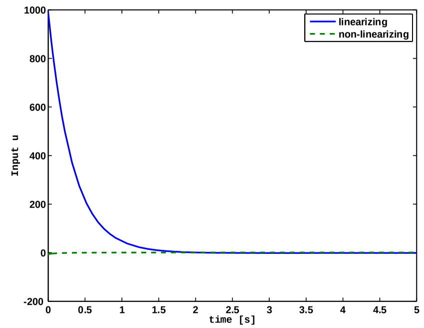
Simulating the two controllers

Simulation with $x(0) = 10$

State trajectory



Control input



The linearizing control is slower and uses excessive input. Thus, linearization can have a significant cost!

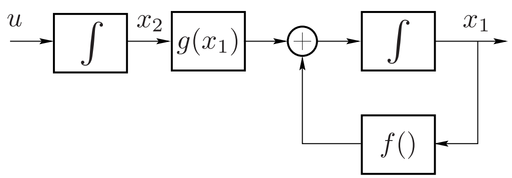
Back-Stepping Control Design

We want to design a state feedback $u = u(x)$ that stabilizes

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= u \end{aligned} \qquad (1)$$

at $x = 0$ with $f(0) = 0$.

Idea: See the system as a cascade connection. Design controller first for the inner loop and then for the outer.



Suppose the partial system

$$\dot{x}_1 = f(x_1) + g(x_1)\bar{v}$$

can be stabilized by $\bar{v} = \phi(x_1)$ and there exists Lyapunov fcn $V_1 = V_1(x_1)$ such that

$$\dot{V}_1(x_1) = \frac{dV_1}{dx_1}\Big(f(x_1) + g(x_1)\phi(x_1)\Big) \le -W(x_1)$$

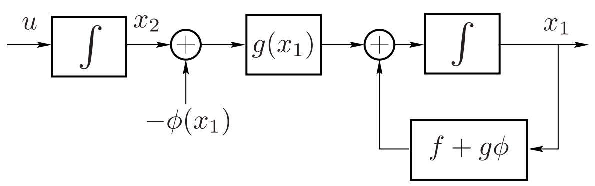
for some positive definite function W.

This is a critical assumption in backstepping control!

The Trick

Equation (1) can be rewritten as

$$\dot{x}_1 = f(x_1) + g(x_1)\phi(x_1) + g(x_1)[x_2 - \phi(x_1)]$$
$$\dot{x}_2 = u$$

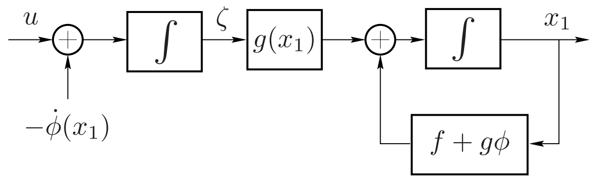


Introduce new state $\zeta = x_2 - \phi(x_1)$ and control $v = u - \dot{\phi}$:

$$\dot{x}_1 = f(x_1) + g(x_1)\phi(x_1) + g(x_1)\zeta$$
$$\dot{\zeta} = v$$

where

$$\dot{\phi}(x_1) = \frac{d\phi}{dx_1}\dot{x}_1 = \frac{d\phi}{dx_1}\Big(f(x_1) + g(x_1)x_2\Big)$$



Consider $V_2(x_1, x_2) = V_1(x_1) + \zeta^2/2$. Then,

$$\dot{V}_2(x_1, x_2) = \frac{dV_1}{dx_1}\Big(f(x_1) + g(x_1)\phi(x_1)\Big) + \frac{dV_1}{dx_1}g(x_1)\zeta + \zeta v$$
$$\le -W(x_1) + \frac{dV_1}{dx_1}g(x_1)\zeta + \zeta v$$

Choosing

$$v = -\frac{dV_1}{dx_1}g(x_1) - k\zeta, \qquad k > 0$$

gives

$$\dot{V}_2(x_1, x_2) \le -W(x_1) - k\zeta^2$$

Hence, $x = 0$ is asymptotically stable for (1) with control law $u(x) = \dot{\phi}(x) + v(x)$.

If V_1 radially unbounded, then global stability.

Back-Stepping Lemma

Lemma: Let $z = (x_1, \dots, x_{k-1})^T$ and

$$\dot{z} = f(z) + g(z)x_k$$
$$\dot{x}_k = u$$

Assume $\phi(0) = 0$, $f(0) = 0$,

$$\dot{z} = f(z) + g(z)\phi(z)$$

stable, and $V(z)$ a Lyapunov fcn (with $\dot{V} \le -W$). Then,

$$u = \frac{d\phi}{dz}\Big(f(z) + g(z)x_k\Big) - \frac{dV}{dz}g(z) - (x_k - \phi(z))$$

stabilizes $x = 0$ with $V(z) + (x_k - \phi(z))^2/2$ being a Lyapunov fcn.

Strict Feedback Systems

Back-stepping Lemma can be applied to stabilize systems on strict feedback form:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2\\ \dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)x_3\\ \dot{x}_3 &= f_3(x_1,x_2,x_3) + g_3(x_1,x_2,x_3)x_4\\ &\vdots\\ \dot{x}_n &= f_n(x_1,\dots,x_n) + g_n(x_1,\dots,x_n)u\end{aligned}$$

where $g_k \neq 0$

Note: $x_1,\dots,x_k$ do not depend on $x_{k+2},\dots,x_n$.

2 minute exercise: *Give an example of a linear system* $\dot{x} = Ax + Bu$ *on strict feedback form.*

Back-Stepping

Back-Stepping Lemma can be applied **recursively** to a system

$$\dot{x} = f(x) + g(x)u$$

on strict feedback form.

Back-stepping generates stabilizing feedbacks $\phi_k(x_1,\dots,x_k)$ (equal to u in Back-Stepping Lemma) and Lyapunov functions

$$V_k(x_1,\dots,x_k) = V_{k-1}(x_1,\dots,x_{k-1}) + [x_k - \phi_{k-1}]^2/2$$

by "stepping back" from x_1 to u (see Khalil pp. 593–594 for details).

Back-stepping results in the final state feedback

$$u = \phi_n(x_1,\dots,x_n)$$

Example

Design back-stepping controller for

$$\dot{x}_1 = x_1^2 + x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

Step 0 Verify strict feedback form
Step 1 Consider first subsystem

$$\dot{x}_1 = x_1^2 + \phi_1(x_1), \quad \dot{x}_2 = u_1$$

where $\phi_1(x_1) = -x_1^2 - x_1$ stabilizes the first equation. With $V_1(x_1) = x_1^2/2$, Back-Stepping Lemma gives

$$u_1 = (-2x_1 - 1)(x_1^2 + x_2) - x_1 - (x_2 + x_1^2 + x_1) = \phi_2(x_1, x_2)$$
$$V_2 = x_1^2/2 + (x_2 + x_1^2 + x_1)^2/2$$

Step 2 Applying Back-Stepping Lemma on

$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\end{aligned}$$

gives

$$\begin{aligned}u = u_2 &= \frac{d\phi_2}{dz}\Big(f(z) + g(z)x_n\Big) - \frac{dV_2}{dz}g(z) - (x_n - \phi_2(z))\\ &= \frac{\partial\phi_2}{\partial x_1}(x_1^2 + x_2) + \frac{\partial\phi_2}{\partial x_2}x_3 - \frac{\partial V_2}{\partial x_2} - (x_3 - \phi_2(x_1, x_2))\end{aligned}$$

which globally stabilizes the system.

EL2620 Nonlinear Control

Lecture 11

- Nonlinear controllability
- Gain scheduling

Today's Goal

You should be able to

- Determine if a nonlinear system is controllable
- Apply gain scheduling to simple examples

Controllability

Definition:

$$\dot{x} = f(x, u)$$

is **controllable** if for any x^0, x^1 there exists $T > 0$ and $u : [0, T] \to \mathbb{R}$ such that $x(0) = x^0$ and $x(T) = x^1$.

Linear Systems

Lemma:

$$\dot{x} = Ax + Bu$$

is controllable if and only if

$$W_n = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

has full rank.

Is there a corresponding result for nonlinear systems?

Controllable Linearization

Lemma: Let

$$\dot{z} = Az + Bu$$

be the linearization of

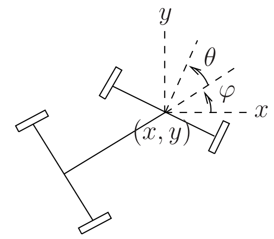
$$\dot{x} = f(x) + g(x)u$$

at $x = 0$ with $f(0) = 0$. If the linear system is controllable then the nonlinear system is controllable in a neighborhood of the origin.

Remark:

- Hence, if $\operatorname{rank} W_n = n$ then there is an $\epsilon > 0$ such that for every $x_1 \in B_\epsilon(0)$ there exists $u : [0, T] \to \mathbb{R}$ so that $x(T) = x_1$
- A nonlinear system can be controllable, even if the linearized system is not controllable

Car Example



Input: u_1 steering wheel velocity, u_2 forward velocity

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ \varphi \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_1 + \begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} u_2 = g_1(z)u_1 + g_2(z)u_2$$

Linearization for $u_1 = u_2 = 0$ gives

$$\dot{z} = Az + B_1 u_1 + B_2 u_2$$

with $A = 0$ and

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \cos(\varphi_0+\theta_0) \\ \sin(\varphi_0+\theta_0) \\ \sin(\theta_0) \\ 0 \end{pmatrix}$$

$\operatorname{rank} W_n = \operatorname{rank}\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = 2 < 4$, so the linearization is not controllable. Still the car is controllable!

Linearization does not capture the controllability good enough

Lie Brackets

Lie bracket between vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field defined by

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

Example:

$$f = \begin{pmatrix} \cos x_2 \\ x_1 \end{pmatrix}, \qquad g = \begin{pmatrix} x_1 \\ 1 \end{pmatrix}$$

$$[f, g] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos x_2 \\ x_1 \end{pmatrix} - \begin{pmatrix} 0 & -\sin x_2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos x_2 + \sin x_2 \\ -x_1 \end{pmatrix}$$

Lie Bracket Direction

For the system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

the control

$$(u_1, u_2) = \begin{cases} (1,0), & t \in [0,\epsilon) \\ (0,1), & t \in [\epsilon, 2\epsilon) \\ (-1,0), & t \in [2\epsilon, 3\epsilon) \\ (0,-1), & t \in [3\epsilon, 4\epsilon) \end{cases}$$

gives motion

$$x(4\epsilon) = x(0) + \epsilon^2 [g_1, g_2] + O(\epsilon^3)$$

The system can move in the $[g_1, g_2]$ direction!

Proof

1. For $t \in [0, \epsilon]$, assuming ϵ small and $x(0) = x_0$, Taylor series yields

$$x(\epsilon) = x_0 + g_1(x_0)\epsilon + \frac{1}{2}\frac{dg_1}{dx} g_1(x_0)\epsilon^2 + \mathcal{O}(\epsilon^3) \qquad (1)$$

2. Similarily, for $t \in [\epsilon, 2\epsilon]$

$$x(2\epsilon) = x(\epsilon) + g_2(x(\epsilon))\epsilon + \frac{1}{2}\frac{dg_2}{dx} g_2(x(\epsilon))\epsilon^2$$

and with $x(\epsilon)$ from (1), and $g_2(x(\epsilon)) = g_2(x_0) + \frac{dg_2}{dx}\epsilon g_1(x_0)$

$$x(2\epsilon) = x_0 + \epsilon(g_1(x_0) + g_2(x_0)) + \epsilon^2 \left(\frac{1}{2}\frac{dg_1}{dx}(x_0) g_1(x_0) + \frac{dg_2}{dx}(x_0) g_1(x_0) + \frac{1}{2}\frac{dg_2}{dx}(x_0) g_2(x_0) \right)$$

Proof, continued

3. Similarily, for $t \in [2\epsilon, 3\epsilon]$

$$x(3\epsilon) = x_0 + \epsilon g_2 + \epsilon^2 \left(\frac{dg_2}{dx} g_1 - \frac{dg_1}{dx} g_2 + \frac{1}{2}\frac{dg_2}{dx} g_2 \right)$$

4. Finally, for $t \in [3\epsilon, 4\epsilon]$

$$x(4\epsilon) = x_0 + \epsilon^2 \left(\frac{dg_2}{dx} g_1 - \frac{dg_1}{dx} g_2 \right)$$

Car Example (Cont'd)

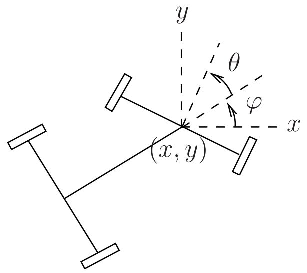
$$g_3 := [g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2$$

$$= \begin{pmatrix} 0 & 0 & -\sin(\varphi+\theta) & -\sin(\varphi+\theta) \\ 0 & 0 & \cos(\varphi+\theta) & \cos(\varphi+\theta) \\ 0 & 0 & 0 & \cos(\theta) \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - 0$$

$$= \begin{pmatrix} -\sin(\varphi+\theta) \\ \cos(\varphi+\theta) \\ \cos(\theta) \\ 0 \end{pmatrix}$$

We can hence move the car in the g_3 direction ("wriggle") by applying the control sequence

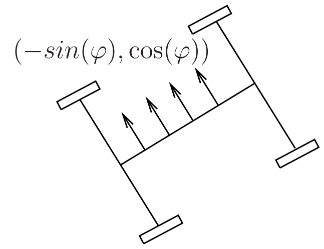
$$(u_1, u_2) = \{(1,0), (0,1), (-1,0), (0,-1)\}$$



The car can also move in the direction

$$g_4 := [g_3, g_2] = \frac{\partial g_2}{\partial x} g_3 - \frac{\partial g_3}{\partial x} g_2 = \ldots = \begin{pmatrix} -\sin(\varphi + 2\theta) \\ \cos(\varphi + 2\theta) \\ 0 \\ 0 \end{pmatrix}$$

g_4 direction corresponds to sideways movement



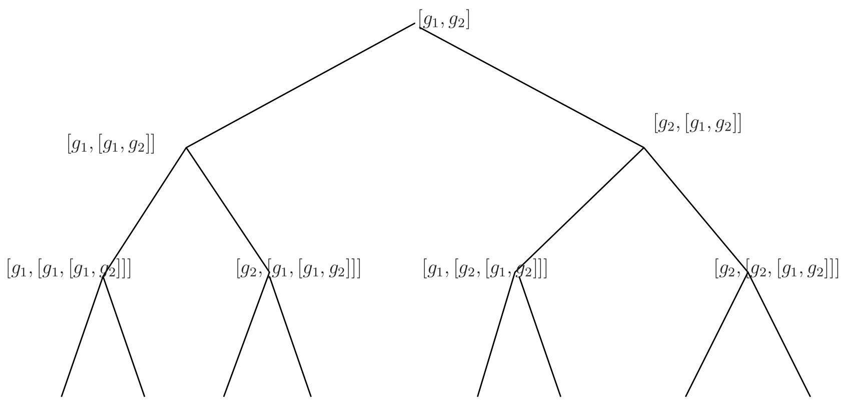
Parking Theorem

You can get out of any parking lot that is $\epsilon > 0$ bigger than your car by applying control corresponding to g_4, that is, by applying the control sequence

Wriggle, Drive, $-$Wriggle, $-$Drive

2 minute exercise: *What does the direction $[g_1, g_2]$ correspond to for a linear system $\dot{x} = g_1(x)u_1 + g_2(x)u_2 = B_1u_1 + B_2u_2$?*

The Lie Bracket Tree



Controllability Theorem

Theorem: The system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

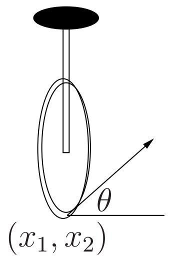
is controllable if the Lie bracket tree (together with g_1 and g_2) spans $\mathbb{R}^n$ for all x

Remark:

- The system can be steered in any direction of the Lie bracket tree

Example—Unicycle

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2$$



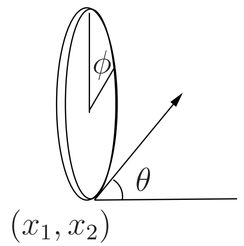
$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad [g_1, g_2] = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$

Controllable because $\{g_1, g_2, [g_1, g_2]\}$ spans $\mathbb{R}^3$

2 minute exercise:

- *Show that* $\{g_1, g_2, [g_1, g_2]\}$ *spans* $\mathbb{R}^3$ *for the unicycle*
- *Is the linearization of the unicycle controllable?*

Example—Rolling Penny



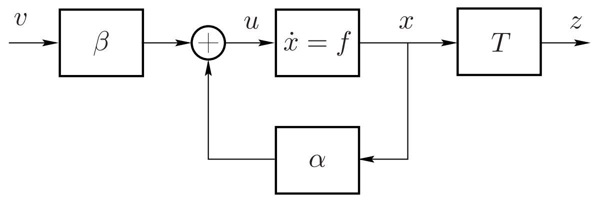
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ 1 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_2$$

Controllable because $\{g_1, g_2, [g_1, g_2], [g_2, [g_1, g_2]]\}$ spans $\mathbb{R}^4$

When is Feedback Linearization Possible?

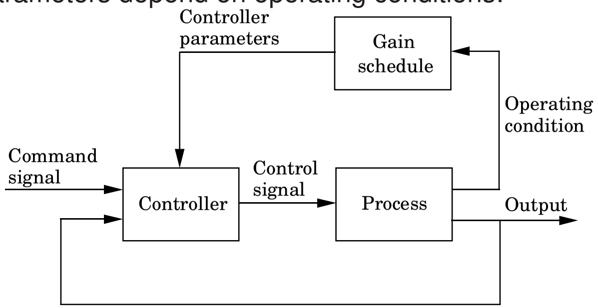
Q: When can we transform $\dot{x} = f(x) + g(x)u$ into $\dot{z} = Az + bv$ by means of feedback $u = \alpha(x) + \beta(x)v$ and change of variables $z = T(x)$ (see previous lecture)?

A: The answer requires Lie brackets and further concepts from differential geometry (see Khalil and PhD course)



Gain Scheduling

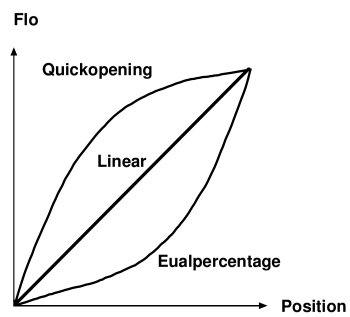
Control parameters depend on operating conditions:



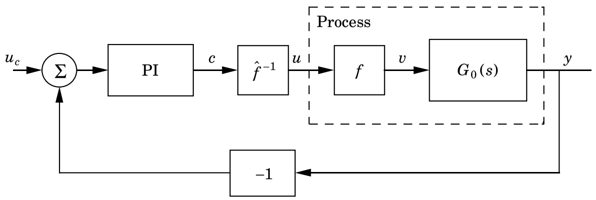
Example: PID controller with $K = K(\alpha)$, where α is the scheduling variable.

Examples of scheduling variable are production rate, machine speed, Mach number, flow rate

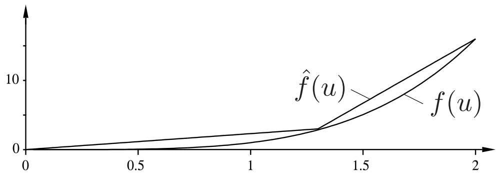
Valve Characteristics



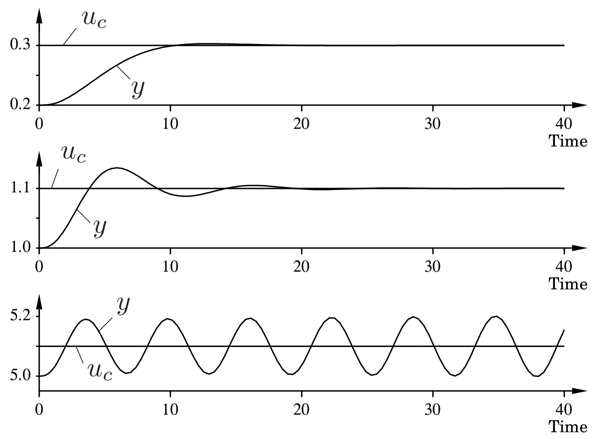
Nonlinear Valve



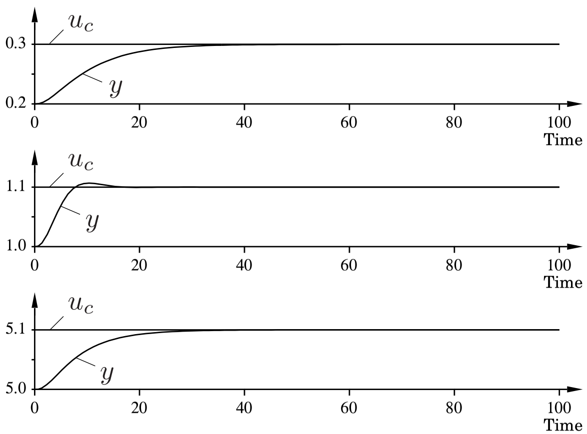
Valve characteristics



Without gain scheduling:

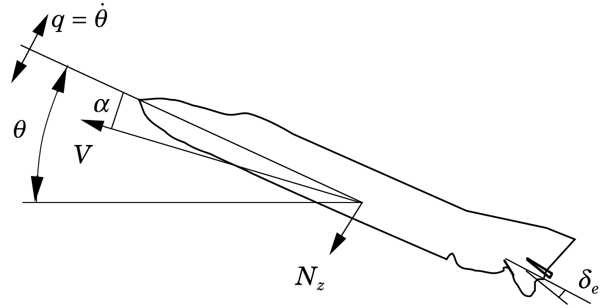


With gain scheduling:



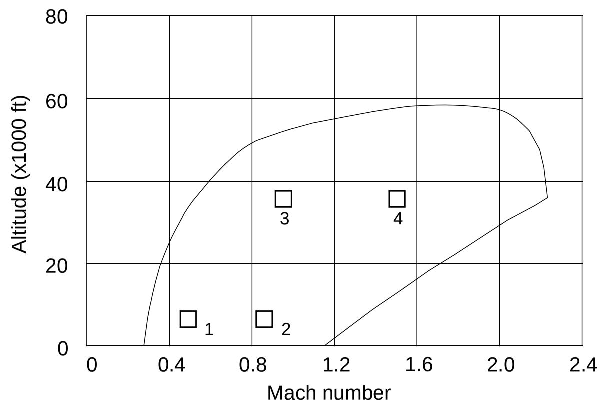
Flight Control

Pitch dynamics

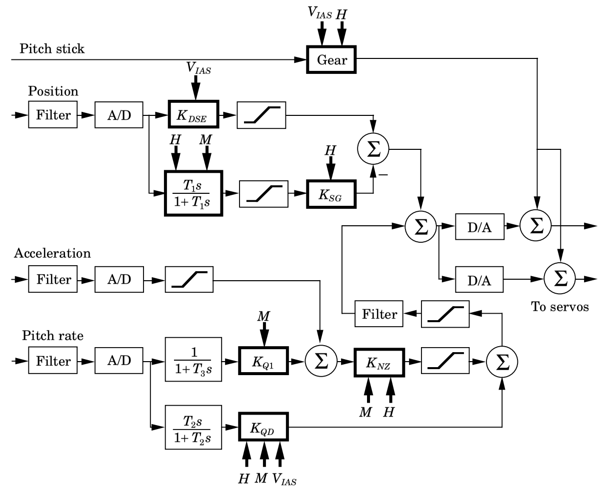


Flight Control

Operating conditions:



The Pitch Control Channel



Today's Goal

You should be able to

- Determine if a nonlinear system is controllable
- Apply gain scheduling to simple examples

EL2620 Nonlinear Control

Lecture 12

- Optimal control

Today's Goal

You should be able to

- Design controllers based on optimal control theory

Optimal Control Problems

Idea: formulate the control design problem as an optimization problem

$$\min_{u(t)} J(x, u, t), \quad \dot{x} = f(t, x, u)$$

+ provides a systematic design framework
+ applicable to nonlinear problems
+ can deal with constraints
- difficult to formulate control objectives as a single objective function
- determining the optimal controller can be hard

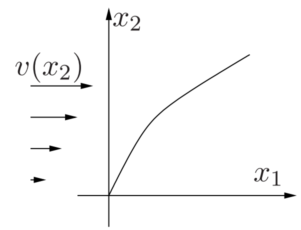
Example—Boat in Stream

Sail as far as possible in x_1 direction

Speed of water $v(x_2)$ with $dv/dx_2 = 1$

Rudder angle control:

$u(t) \in U = \{(u_1, u_2) : u_1^2 + u_2^2 = 1\}$



$$\max_{u:[0,t_f]\to U} x_1(t_f)$$

$$\dot{x}_1(t) = v(x_2) + u_1(t)$$

$$\dot{x}_2(t) = u_2(t)$$

$$x_1(0) = x_2(0) = 0$$

Example—Resource Allocation

Maximization of stored profit

$x(t) \in [0, \infty)$	production rate
$u(t) \in [0, 1]$	portion of x reinvested
$1 - u(t)$	portion of x stored
$\gamma u(t)x(t)$	change of production rate ($\gamma > 0$)
$[1 - u(t)]x(t)$	amount of stored profit

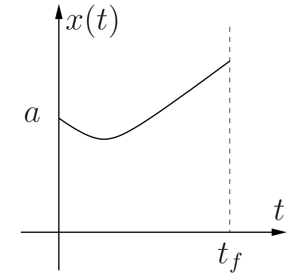
$$\max_{u:[0,t_f]\to[0,1]} \int_0^{t_f} [1 - u(t)]x(t)dt$$
$$\dot{x}(t) = \gamma u(t)x(t)$$
$$x(0) = x_0 > 0$$

Example—Minimal Curve Length

Find the curve with minimal length between a given point and a line

Curve: $(t, x(t))$ with $x(0) = a$

Line: Vertical through $(t_f, 0)$



$$\min_{u:[0,t_f]\to\mathbb{R}} \int_0^{t_f} \sqrt{1 + u^2(t)}dt$$
$$\dot{x}(t) = u(t)$$
$$x(0) = a$$

Optimal Control Problem

Standard form:

$$\min_{u:[0,t_f]\to U} \int_0^{t_f} L(x(t), u(t))\, dt + \phi(x(t_f))$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

Remarks:

- $U \subset \mathbb{R}^m$ set of admissible control
- Infinite dimensional optimization problem:
  Optimization over *functions* $u : [0, t_f] \to U$
- Constraints on x from the dynamics
- Final time t_f fixed (free later)

Pontryagin's Maximum Principle

Theorem: Introduce the Hamiltonian function

$$H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u)$$

Suppose the optimal control problem above has the solution $u^* : [0, t_f] \to U$ and $x^* : [0, t_f] \to \mathbb{R}^n$. Then,

$$\min_{u\in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), \quad \forall t \in [0, t_f]$$

where $\lambda(t)$ solves the adjoint equation

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial x}(x^*(t), u^*(t), \lambda(t)), \quad \lambda(t_f) = \frac{\partial \phi^T}{\partial x}(x^*(t_f))$$

Moreover, the optimal control is given by

$$u^*(t) = \arg\min_{u\in U} H(x^*(t), u, \lambda(t))$$

Remarks

- See textbook, e.g., Glad and Ljung, for proof. The outline is simply to note that every change of $u(t)$ from the optimal $u^*(t)$ must increase the criterium. Then perform a clever Taylor expansion.
- Pontryagin's Maximum Principle provides **necessary** condition: there may exist many or none solutions
  (cf., $\min_{u:[0,1]\to\mathbb{R}} x(1)$, $\dot{x} = u$, $x(0) = 0$)
- The Maximum Principle provides all possible candidates.
- Solution involves $2n$ ODE's with boundary conditions $x(0) = x_0$ and $\lambda(t_f) = \partial\phi^T/\partial x(x^*(t_f))$. Often hard to solve explicitly.
- "maximum" is due to Pontryagin's original formulation

Example—Boat in Stream (cont'd)

Hamiltonian satisfies

$$H = \lambda^T f = \begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}\begin{pmatrix} v(x_2) + u_1 \\ u_2 \end{pmatrix}$$

$$\frac{\partial H}{\partial x} = \begin{pmatrix}0 & \lambda_1\end{pmatrix}, \qquad \phi(x) = -x_1$$

Adjoint equations

$$\dot{\lambda}_1(t) = 0, \qquad \lambda_1(t_f) = -1$$
$$\dot{\lambda}_2(t) = -\lambda_1(t), \qquad \lambda_2(t_f) = 0$$

have solution

$$\lambda_1(t) = -1, \quad \lambda_2(t) = t - t_f$$

Optimal control

$$u^*(t) = \arg \min_{u_1^2+u_2^2=1} \lambda_1(t)(v(x_2^*(t)) + u_1) + \lambda_2(t)u_2$$
$$= \arg \min_{u_1^2+u_2^2=1} \lambda_1(t)u_1 + \lambda_2(t)u_2$$

Hence,

$$u_1(t) = -\frac{\lambda_1(t)}{\sqrt{\lambda_1^2(t) + \lambda_2^2(t)}}, \quad u_2(t) = -\frac{\lambda_2(t)}{\sqrt{\lambda_1^2(t) + \lambda_2^2(t)}}$$

or

$$u_1(t) = \frac{1}{\sqrt{1 + (t - t_f)^2}}, \quad u_2(t) = \frac{t_f - t}{\sqrt{1 + (t - t_f)^2}}$$

Example—Resource Allocation (cont'd)

$$\min_{u:[0,t_f]\to[0,1]} \int_0^{t_f} [u(t) - 1]x(t)dt$$
$$\dot{x}(t) = \gamma u(t)x(t), \qquad x(0) = x_0$$

Hamiltonian satisfies

$$H = L + \lambda^T f = (u - 1)x + \lambda\gamma u x$$

Adjoint equation

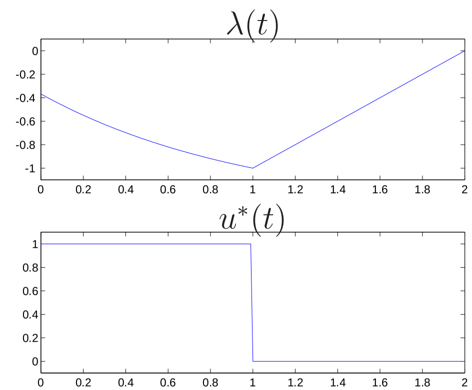
$$\dot{\lambda}(t) = 1 - u^*(t) - \lambda(t)\gamma u^*(t), \qquad \lambda(t_f) = 0$$

Optimal control

$$u^*(t) = \arg\min_{u\in[0,1]} (u-1)x^*(t) + \lambda(t)\gamma u x^*(t)$$
$$= \arg\min_{u\in[0,1]} u(1+\lambda(t)\gamma), \qquad (x^*(t) > 0)$$
$$= \begin{cases} 0, & \lambda(t) \geq -1/\gamma \\ 1, & \lambda(t) < -1/\gamma \end{cases}$$

For $t \approx t_f$, we have $u^*(t) = 0$ (why?) and thus $\dot\lambda(t) = 1$.

For $t < t_f - 1/\gamma$, we have $u^*(t) = 1$ and thus $\dot\lambda(t) = -\gamma\lambda(t)$.



- $u^*(t) = \begin{cases} 1, & t \in [0, t_f - 1/\gamma] \\ 0, & t \in (t_f - 1/\gamma, t_f] \end{cases}$
- It is optimal to reinvest in the beginning

5 minute exercise: Find the curve with minimal length by solving

$$\min_{u:[0,t_f]\to\mathbb{R}} \int_0^{t_f} \sqrt{1+u^2(t)}\,dt$$
$$\dot x(t) = u(t), \qquad x(0) = a$$

5 minute exercise II: Solve the optimal control problem

$$\min \int_0^1 u^4 dt + x(1)$$
$$\dot x = -x + u$$
$$x(0) = 0$$

History—Calculus of Variations

- Brachistochrone (shortest time) problem (1696): Find the (frictionless) curve that takes a particle from A to B in shortest time

$$dt = \frac{ds}{v} = \frac{\sqrt{dx^2 + dy^2}}{v} = \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}} dx$$

Minimize

$$J(y) = \int_A^B \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}} \, dx$$

Solved by John and James Bernoulli, Newton, l'Hospital

- Find the curve enclosing largest area (Euler)

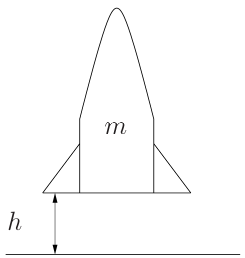
History—Optimal Control

- The space race (Sputnik, 1957)
- Pontryagin's Maximum Principle (1956)
- Bellman's Dynamic Programming (1957)
- Huge influence on engineering and other sciences:
  - Robotics—trajectory generation
  - Aeronautics—satellite orbits
  - Physics—Snell's law, conservation laws
  - Finance—portfolio theory

Goddard's Rocket Problem (1910)

How to send a rocket as high up in the air as possible?

$$\frac{d}{dt}\begin{pmatrix} v \\ h \\ m \end{pmatrix} = \begin{pmatrix} \frac{u - D}{m} - g \\ v \\ -\gamma u \end{pmatrix}$$



$(v(0), h(0), m(0)) = (0, 0, m_0)$, $g, \gamma > 0$

u motor force, $D = D(v, h)$ air resistance

Constraints: $0 \le u \le u_{max}$ and $m(t_f) = m_1$ (empty)

Optimization criterion: $\max_u h(t_f)$

Generalized form:

$$\min_{u:[0,t_f]\to U} \int_0^{t_f} L(x(t), u(t))\, dt + \phi(x(t_f))$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$
$$\psi(x(t_f)) = 0$$

Note the diffences compared to standard form:

- End time t_f is free
- Final state is constrained: $\psi(x(t_f)) = x_3(t_f) - m_1 = 0$

Solution to Goddard's Problem

Goddard's problem is on generalized form with

$$x = (v, h, m)^T, \quad L \equiv 0, \quad \phi(x) = -x_2, \quad \psi(x) = x_3 - m_1$$

$D(v, h) \equiv 0$:

- Easy: let $u(t) = u_{max}$ until $m(t) = m_1$
- Burn fuel as fast as possible, because it costs energy to lift it

$D(v, h) \not\equiv 0$:

- Hard: e.g., it can be optimal to have low speed when air resistance is high, in order to burn fuel at higher level
- Took 50 years before a complete solution was presented

General Pontryagin's Maximum Principle

Theorem: Suppose $u^* : [0, t_f] \to U$ and $x^* : [0, t_f] \to \mathbb{R}^n$ are solutions to

$$\min_{u:[0,t_f]\to U} \int_0^{t_f} L(x(t), u(t))\, dt + \phi(t_f, x(t_f))$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$
$$\psi(t_f, x(t_f)) = 0$$

Then, there exists $n_0 \geq 0$, $\mu \in \mathbb{R}^n$ such that $(n_0, \mu^T) \neq 0$ and

$$\min_{u\in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad t \in [0, t_f]$$

where

$$H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda^T f(x, u)$$

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial x}(x^*(t), u^*(t), \lambda(t), n_0)$$
$$\lambda^T(t_f) = n_0 \frac{\partial \phi}{\partial x}(t_f, x^*(t_f)) + \mu^T \frac{\partial \psi}{\partial x}(t_f, x^*(t_f))$$

$$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0)$$
$$= -n_0 \frac{\partial \phi}{\partial t}(t_f, x^*(t_f)) - \mu^T \frac{\partial \psi}{\partial t}(t_f, x^*(t_f))$$

Remarks:

- t_f may be a free variable
- With fixed t_f: $H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$
- ψ defines end point constraints

Example—Minimum Time Control

Bring the states of the double integrator to the origin as fast as possible

$$\min_{u:[0,t_f]\to[-1,1]} \int_0^{t_f} 1\, dt = \min_{u:[0,t_f]\to[-1,1]} t_f$$
$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t)$$
$$\psi(x(t_f)) = (x_1(t_f), x_2(t_f))^T = (0, 0)^T$$

Optimal control is the bang-bang control

$$u^*(t) = \arg \min_{u\in[-1,1]} 1 + \lambda_1(t) x_2^*(t) + \lambda_2(t) u$$
$$= \begin{cases} 1, & \lambda_2(t) < 0 \\ -1, & \lambda_2(t) \geq 0 \end{cases}$$

Adjoint equations $\dot{\lambda}_1(t) = 0, \quad \dot{\lambda}_2(t) = -\lambda_1(t)$ gives

$$\lambda_1(t) = c_1, \quad \lambda_2(t) = c_2 - c_1 t$$

With $u(t) = \zeta = \pm 1$, we have

$$x_1(t) = x_1(0) + x_2(0)t + \zeta t^2/2$$
$$x_2(t) = x_2(0) + \zeta t$$

Eliminating t gives curves

$$x_1(t) \pm x_2(t)^2/2 = \text{const}$$

These define the *switch curve*, where the optimal control switch

Reference Generation using Optimal Control

- Optimal control problem makes no distinction between open-loop control $u^*(t)$ and closed-loop control $u^*(t,x)$.
- We may use the optimal open-loop solution $u^*(t)$ as the reference value to a linear regulator, which keeps the system close to the wanted trajectory
- Efficient design method for nonlinear problems

Linear Quadratic Control

$$\min_{u:[0,\infty)\to\mathbb{R}^m} \int_0^\infty (x^TQx + u^TRu)\,dt$$

with

$$\dot{x} = Ax + Bu$$

has optimal solution

$$u = -Lx$$

where $L = R^{-1}B^TS$ and $S > 0$ is the solution to

$$SA + A^TS + Q - SBR^{-1}B^TS = 0$$

Properties of LQ Control

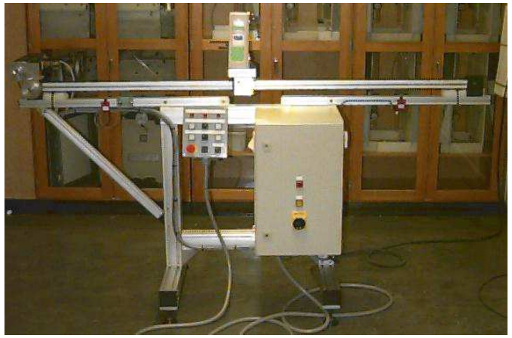
- Stabilizing
- Closed-loop system stable with $u = -\alpha(t)Lx$ for $\alpha(t) \in [1/2, \infty)$ (infinite gain margin)
- Phase margin 60 degrees

If x is not measurable, then one may use a Kalman filter; leads to linear quadratic Gaussian (LQG) control.

- But, then system may have arbitrarily poor robustness! (Doyle, 1978)

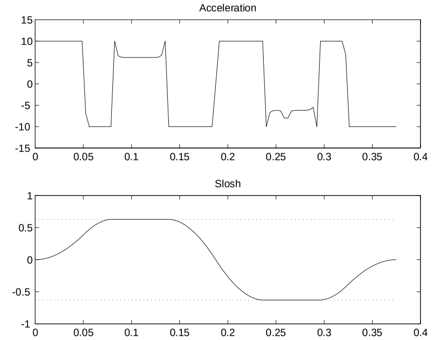
Tetra Pak Milk Race

Move milk in minimum time without spilling



[Grundelius & Bernhardsson,1999]

Given dynamics of system and maximum slosh $\phi = 0.63$, solve $\min_{u:[0,t_f]\to[-10,10]} \int_0^{t_f} 1\, dt$, where u is the acceleration.



Optimal time = 375 ms, TetraPak = 540ms

Pros & Cons for Optimal Control

+ Systematic design procedure
+ Applicable to nonlinear control problems
+ Captures limitations (as optimization constraints)
− Hard to find suitable criteria
− Hard to solve the equations that give optimal controller

SF2852 Optimal Control Theory

- **Period 3, 7.5 credits**
- Optimization and Systems Theory
  `http://www.math.kth.se/optsyst/`

Dynamic Programming: Discrete & continuous; Principle of optimality; Hamilton-Jacobi-Bellman equation

Pontryagin's Maximum principle: Main results; Special cases such as time optimal control and LQ control

Numerical Methods: Numerical solution of optimal control problems

Applications: Aeronautics, Robotics, Process Control, Bioengineering, Economics, Logistics

Today’s Goal

You should be able to

- Design controllers based on optimal control theory for
  - Standard form
  - Generalized form
- Understand possibilities and limitations of optimal control

EL2620 Nonlinear Control

Lecture 13

- Fuzzy logic and fuzzy control
- Artificial neural networks

Some slides copied from K.-E. Årzén and M. Johansson

Today's Goal

You should

- understand the basics of fuzzy logic and fuzzy controllers
- understand simple neural networks

Fuzzy Control

- Many plants are manually controlled by experienced operators
- Transfer process knowledge to control algorithm is difficult

Idea:

- Model operator's control actions (instead of the plant)
- Implement as rules (instead of as differential equations)

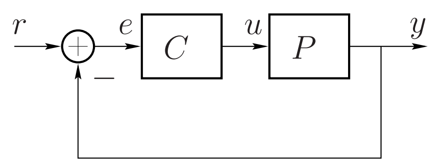
Example of a rule:

```
IF Speed is High AND Traffic is Heavy
   THEN Reduce Gas A Bit
```


Model Controller Instead of Plant

Conventional control design
Model plant $P \rightarrow$ Analyze feedback $\rightarrow$ Synthesize controller $C \rightarrow$ Implement control algorithm

Fuzzy control design
Model manual control $\rightarrow$ Implement control rules



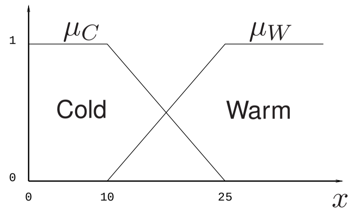
Fuzzy Set Theory

Specify how well an object satisfies a (vague) description

Conventional set theory: $x \in A$ or $x \notin A$

Fuzzy set theory: $x \in A$ to a certain degree $\mu_A(x)$

Membership function:
$\mu_A : \Omega \to [0,1]$ expresses the degree x belongs to A

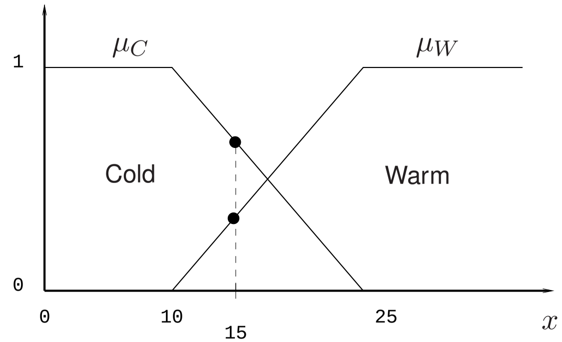


A **fuzzy set** is defined as (A, μ_A)

Example

Q1: Is the temperature $x = 15$ cold?
A1: It is quite cold since $\mu_C(15) = 2/3$.

Q2: Is $x = 15$ warm?
A2: It is not really warm since $\mu_W(15) = 1/3$.



Fuzzy Logic

How to calculate with fuzzy sets (A, μ_A)?

Conventional logic:
AND: $A \cap B$
OR: $A \cup B$
NOT: A'

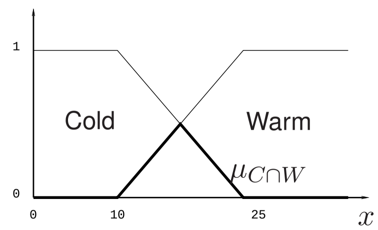
Fuzzy logic:
AND: $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$
OR: $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$
NOT: $\mu_{A'}(x) = 1 - \mu_A(x)$

Defines logic calculations as X AND Y OR Z

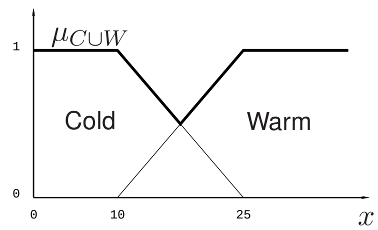
Mimic human linguistic (approximate) reasoning [Zadeh,1965]

Example

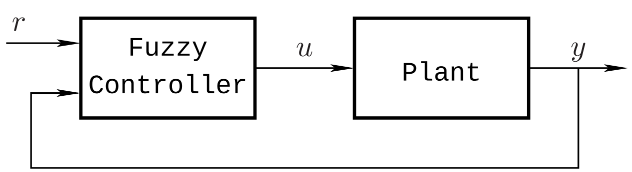
Q1: Is it cold AND warm?



Q2: Is it cold OR warm?



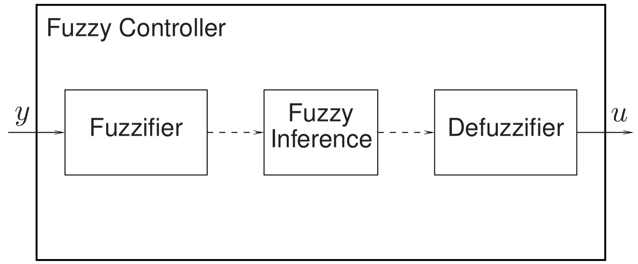
Fuzzy Control System



$r, y, u : [0, \infty) \mapsto \mathbb{R}$ are conventional signals

Fuzzy controller is a nonlinear mapping from y (and r) to u

Fuzzy Controller



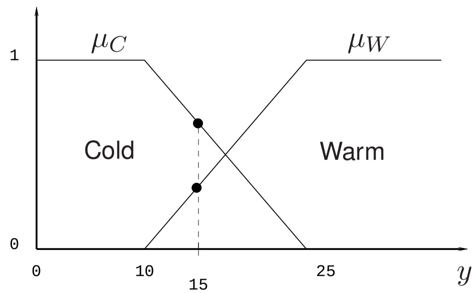
Fuzzifier: Fuzzy set evaluation of y (and r)
Fuzzy Inference: Fuzzy set calculations
Defuzzifier: Map fuzzy set to u

Fuzzifier and defuzzifier act as interfaces to the crisp signals

Fuzzifier

Fuzzy set evaluation of input y

Example
$y = 15$: $\mu_C(15) = 2/3$ and $\mu_W(15) = 1/3$



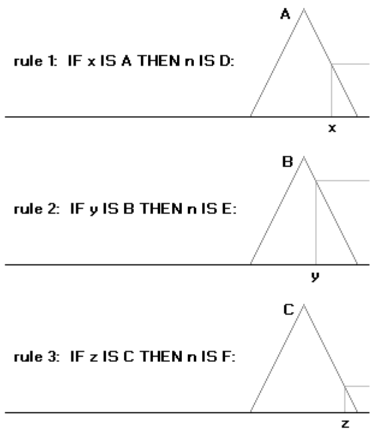
Fuzzy Inference

Fuzzy Inference:

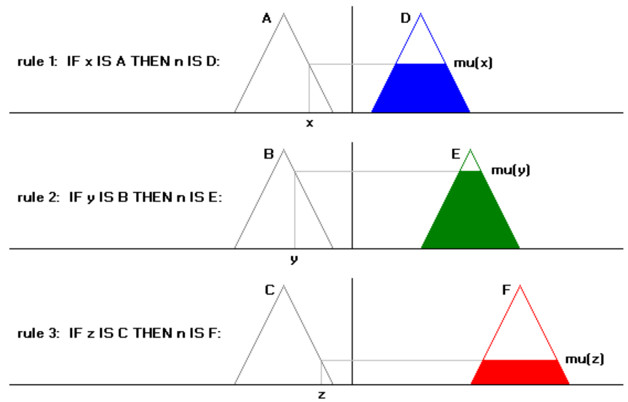
1. Calculate degree of fulfillment for each rule
2. Calculate fuzzy output of each rule
3. Aggregate rule outputs

Examples of fuzzy rules:
Rule 1: IF $\underbrace{y \text{ is Cold}}_{1.}$ THEN $\underbrace{u \text{ is High}}_{2.}$
Rule 2: IF $\underbrace{y \text{ is Warm}}_{1.}$ THEN $\underbrace{u \text{ is Low}}_{2.}$

1. Calculate degree of fulfillment for rules

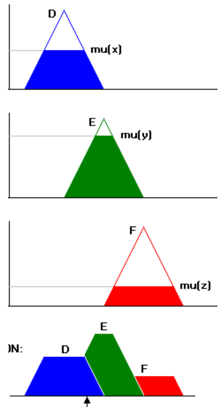


2. Calculate fuzzy output of each rule

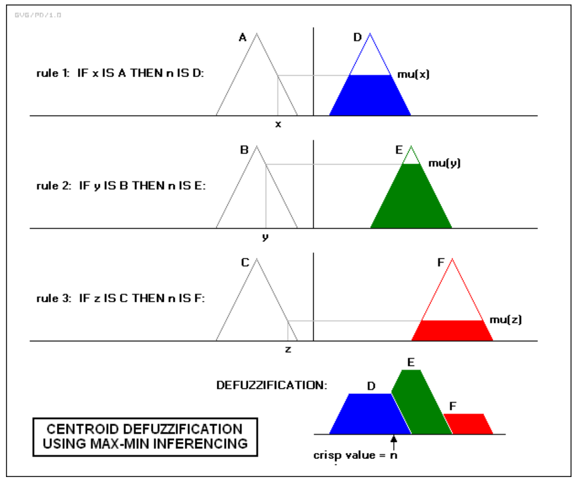


Note that "mu" is standard fuzzy-logic nomenclature for "truth value":

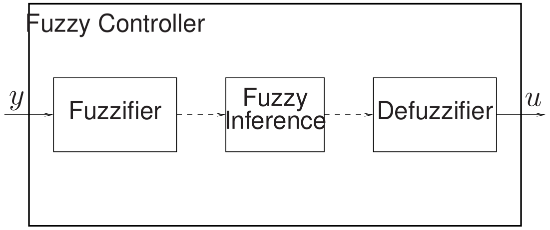
3. Aggregate rule outputs



Defuzzifier



Fuzzy Controller—Summary



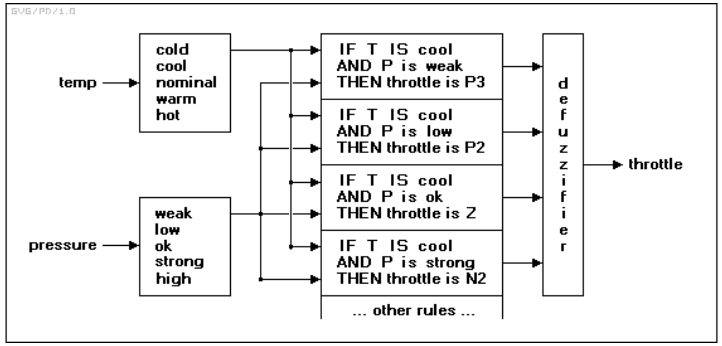
Fuzzifier: Fuzzy set evaluation of y (and r)

Fuzzy Inference: Fuzzy set calculations

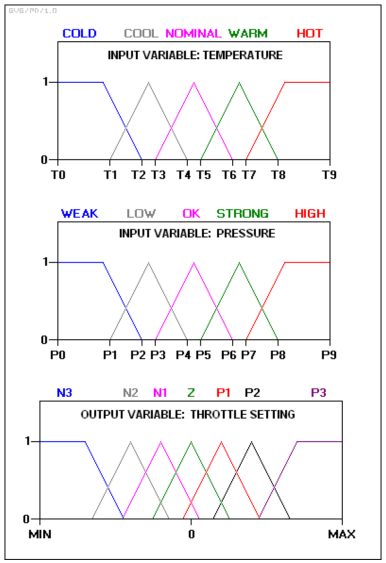
1. Calculate degree of fulfillment for each rule
2. Calculate fuzzy output of each rule
3. Aggregate rule outputs

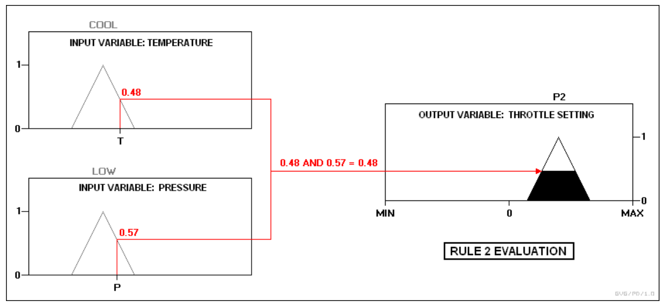
Defuzzifier: Map fuzzy set to u

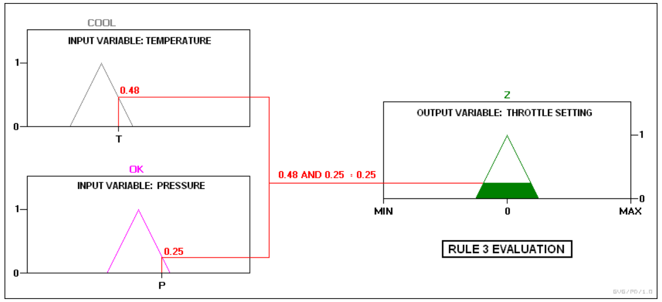
Example—Fuzzy Control of Steam Engine

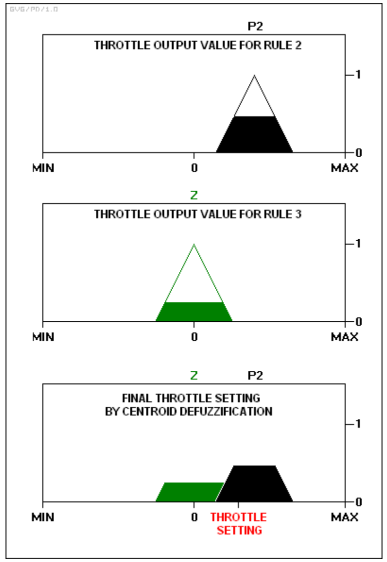


http://isc.faqs.org/docs/air/ttfuzzy.html

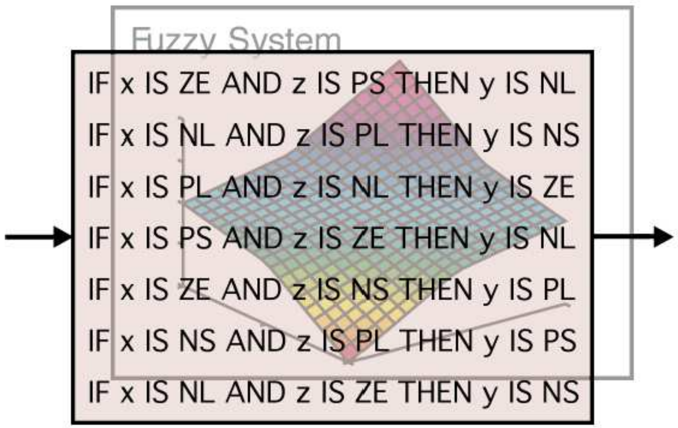




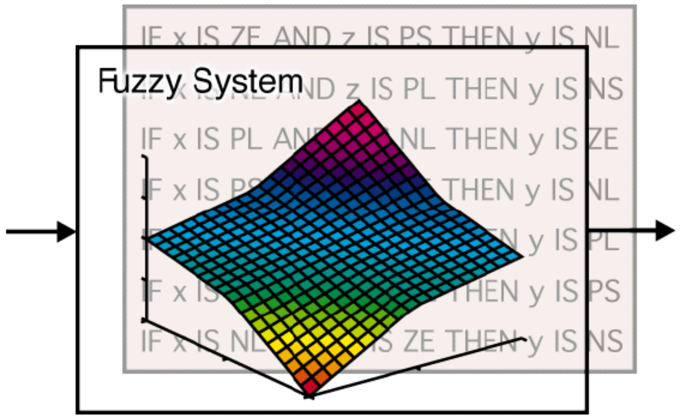




Rule-Based View of Fuzzy Control



Nonlinear View of Fuzzy Control



Pros and Cons of Fuzzy Control

Advantages

- User-friendly way to design nonlinear controllers
- Explicit representation of operator (process) knowledge
- Intuitive for non-experts in conventional control

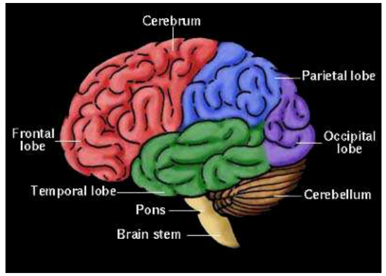
Disadvantages

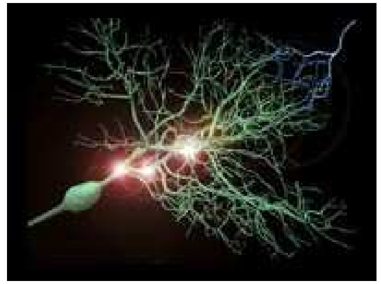
- Limited analysis and synthesis
- Sometimes hard to combine with classical control
- Not obvious how to include dynamics in controller

Fuzzy control is a way to obtain a class of nonlinear controllers

Neural Networks

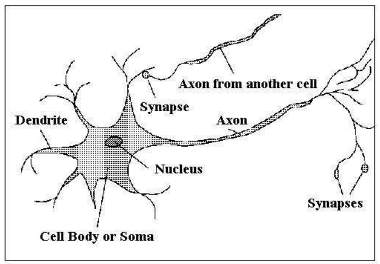
- How does the brain work?
- A network of computing components (neurons)



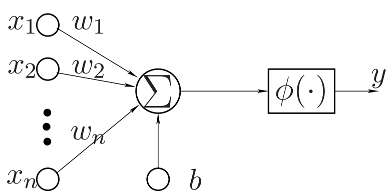


Neurons

Brain neuron



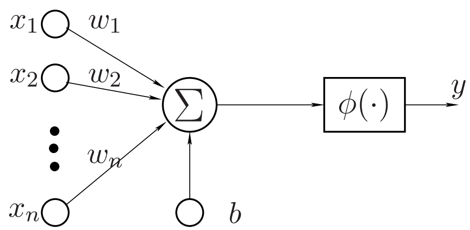
Artificial neuron



Model of a Neuron

Inputs: $x_1, x_2, \ldots, x_n$
Weights: $w_1, w_2, \ldots, w_n$
Bias: b
Nonlinearity: $\phi(\cdot)$
Output: y

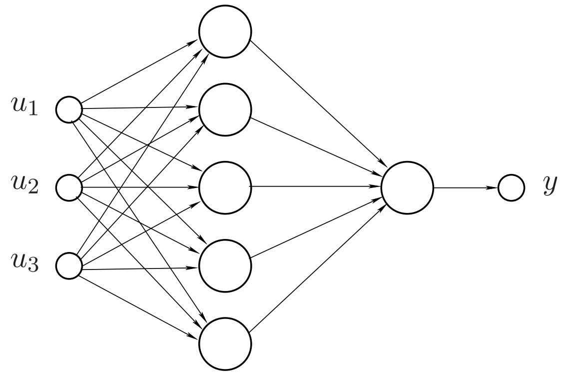
$$y = \phi\big(b + \textstyle\sum_{i=1}^{n} w_i x_i\big)$$



A Simple Neural Network

Neural network consisting of six neurons:

Input Layer Hidden Layer Output Layer



Represents a nonlinear mapping from inputs to outputs

Neural Network Design

1. How many hidden layers?
2. How many neurons in each layer?
3. How to choose the weights?

The choice of weights are often done adaptively through **learning**

Success Stories

Fuzzy controls:

- Zadeh (1965)
- Complex problems but with possible linguistic controls
- Applications took off in mid 70's
  - Cement kilns, washing machines, vacuum cleaners

Artificial neural networks:

- McCulloch & Pitts (1943), Minsky (1951)
- Complex problems with unknown and highly nonlinear structure
- Applications took off in mid 80's
  - Pattern recognition (e.g., speech, vision), data classification

Today's Goal

You should

- understand the basics of fuzzy logic and fuzzy controllers
- understand simple neural networks

Next Lecture

- EL2620 Nonlinear Control revisited
- Spring courses in control
- Master thesis projects
- PhD thesis projects

EL2620 Nonlinear Control

Lecture 14

- Summary and repetition
- Spring courses in control
- Master thesis projects

Exam

- Regular written exam (in English) with five problems
- Sign up on course homepage
- You may bring lecture notes, Glad & Ljung “Reglerteknik”, and TEFYMA or BETA
  (No other material: textbooks, exercises, calculators etc. Any other basic control book must be approved by me *before* the exam.).
- See homepage for old exams

Question 1

What’s on the exam?

- Nonlinear models: equilibria, phase portaits, linearization and stability
- Lyapunov stability (local and global), LaSalle
- Circle Criterion, Small Gain Theorem, Passivity Theorem
- Compensating static nonlinearities
- Describing functions
- Sliding modes, equivalent controls
- Lyapunov based design: back-stepping
- Exact feedback linearization, input-output linearization, zero dynamics
- Nonlinear controllability
- Optimal control

Question 2

What design method should I use in practice?

The answer is highly problem dependent. Possible (learning) approach:

- Start with the simplest:
  - linear methods (loop shaping, state feedback, . . .)
- Evaluate:
  - strong nonlinearities (under feedback!)?
  - varying operating conditions?
  - analyze and simulate with nonlinear model
- Some nonlinearities to compensate for?
  - saturations, valves etc
- Is the system generically nonlinear? E.g, $\dot{x} = xu$

Question 3

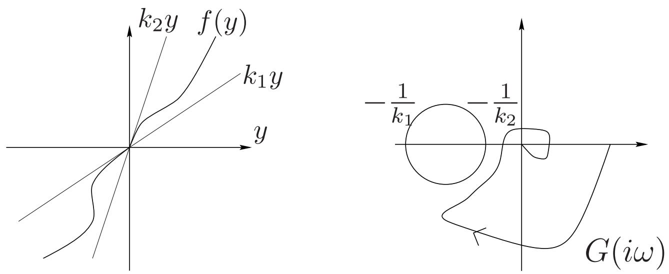
Can a system be proven stable with the Small Gain Theorem and unstable with the Circle Criterion?

- No, the Small Gain Theorem, Passivity Theorem and Circle Criterion all provide only **sufficient conditions for stability**
- But, if one method does not prove stability, another one may.
- Since they do not provide necessary conditions for stability, none of them can be used to prove instability.

Question 4

Can you review the circle criterion? What about $k_1 < 0 < k_2$?

The Circle Criterion



Theorem Consider a feedback loop with $y = Gu$ and $u = -f(y)$. Assume $G(s)$ is stable and that

$$k_1 \le f(y)/y \le k_2.$$

If the Nyquist curve of $G(s)$ stays on the correct side of the circle defined by the points $-1/k_1$ and $-1/k_2$, then the closed-loop system is BIBO stable.

The different cases

Stable system G

1. $0 < k_1 < k_2$: Stay outside circle
2. $0 = k_1 < k_2$: Stay to the right of the line Re $s = -1/k_2$
3. $k_1 < 0 < k_2$: Stay inside the circle

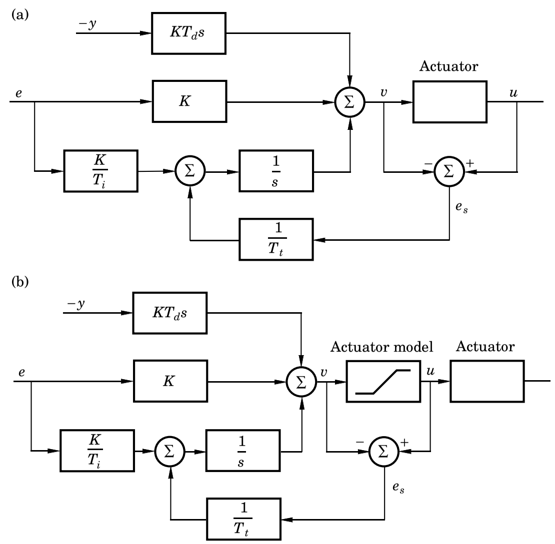
Other cases: Multiply f and G with -1.

Only Case 1 and 2 studied in lectures. Only G stable studied.

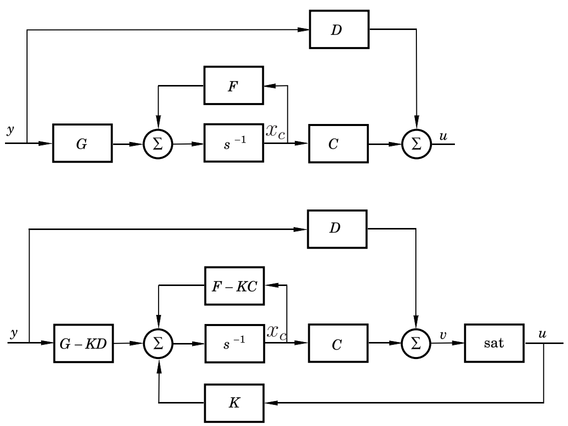
Question 5

Please repeat antiwindup

Tracking PID



Antiwindup—General State-Space Model



Choose K such that $F - KC$ has stable eigenvalues.

Question 6

Please repeat Lyapunov theory

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

locally stable, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

locally asymptotically stable, if locally stable and

$$\|x(0)\| < r \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0$$

globally asymptotically stable, if asymptotically stable for all $x(0) \in \mathbf{R}^n$.

Lyapunov Theorem for Local Stability

Theorem Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$. Assume that $V : \Omega \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
- $\dot{V}(x) \leq 0$ along all trajectories in Ω

then $x = 0$ is locally stable. Furthermore, if

- $\dot{V}(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

Lyapunov Theorem for Global Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$. Assume that $V : \mathbf{R}^n \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \neq 0$
- $\dot{V}(x) < 0$ for all $x \neq 0$
- $V(x) \to \infty$ as $\|x\| \to \infty$

  then $x = 0$ is globally asymptotically stable.

LaSalle's Theorem for Global Asymptotic Stability

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $\mathbb{C}^1$ function $V : R^n \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \neq 0$

(3) $\dot{V}(x) \leq 0$ for all x

(4) $V(x) \to \infty$ as $\|x\| \to \infty$

(5) The only solution of $\dot{x} = f(x)$ such that $\dot{V}(x) = 0$ is $x(t) = 0$ for all t

then $x = 0$ is globally asymptotically stable.

LaSalle's Invariant Set Theorem

Theorem Let $\Omega \in \mathbf{R}^n$ be a bounded and closed set that is invariant with respect to

$$\dot{x} = f(x).$$

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a C^1 function such that $\dot{V}(x) \leq 0$ for $x \in \Omega$. Let E be the set of points in Ω where $\dot{V}(x) = 0$. If M is the largest invariant set in E, then every solution with $x(0) \in \Omega$ approaches M as $t \to \infty$

Remark: a **compact set** (bounded and closed) is obtained if we e.g., consider

$$\Omega = \{x \in \mathbf{R}^n | V(x) \leq c\}$$

and V is a positive definite function

Relation to Poincare-Bendixson Theorem

Poincare-Bendixson *Any orbit of a continuous 2nd order system that stays in a compact region of the phase plane approaches its ω-limit set, which is either a fixed point, a periodic orbit, or several fixed points connected through homoclinic or heteroclinic orbits*

In particular, if the compact region does not contain any fixed point then the ω-limit set is a limit cycle

Example: Pendulum with friction

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

$$V(x) = \frac{g}{l}(1-\cos x_1) + \frac{1}{2}x_2^2 \quad \Rightarrow \quad \dot{V} = -\frac{k}{m}x_2^2$$

- We can not prove global asymptotic stability; why?
- The set $E = \{(x_1, x_2) | \dot{V} = 0\}$ is $E = \{(x_1, x_2) | x_2 = 0\}$
- The invariant points in E are given by $\dot{x}_1 = x_2 = 0$ and $\dot{x}_2 = 0$. Thus, the largest invariant set in E is

$$M = \{(x_1, x_2) | x_1 = k\pi, x_2 = 0\}$$

- The domain is compact if we consider $\Omega = \{(x_1, x_2) \in \mathbf{R}^2 | V(x) \leq c\}$

- If we e.g., consider $\Omega : x_1^2 + x_2^2 \leq 1$ then $M = \{(x_1, x_2) | x_1 = 0, x_2 = 0\}$ and we have proven asymptotic stability of the origin.

Question 7

Please repeat the most important facts about sliding modes.

There are 3 essential parts you need to understand:

1. The sliding manifold
2. The sliding control
3. The equivalent control

Step 1. The Sliding Manifold S

Aim: we want to stabilize the equilibrium of the dynamic system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^1$$

Idea: use u to force the system onto a *sliding manifold* S of dimension $n-1$ in finite time

$$S = \{x \in \mathbb{R}^n | \sigma(x) = 0\} \quad \sigma \in R^1$$

and make S invariant

If $x \in \mathbb{R}^2$ then S is $\mathbb{R}^1$, i.e., a curve in the state-plane (phase plane).

Example

$$\dot{x}_1 = x_2(t)$$
$$\dot{x}_2 = x_1(t)x_2(t) + u(t)$$

Choose S for desired behavior, e.g.,

$$\sigma(x) = ax_1 + x_2 = 0 \quad \Rightarrow \quad \dot{x}_1 = -ax_1(t)$$

Choose large a: fast convergence along sliding manifold

Step 2. The Sliding Controller

Use Lyapunov ideas to design $u(x)$ such that S is an attracting invariant set

Lyapunov function $V(x) = 0.5\sigma^2$ yields $\dot{V} = \sigma\dot{\sigma}$

For 2nd order system $\dot{x}_1 = x_2$, $\dot{x}_2 = f(x) + g(x)u$ and $\sigma = x_1 + x_2$ we get

$$\dot{V} = \sigma\left(x_2 + f(x) + g(x)u\right) < 0 \quad \Leftarrow \quad u = -\frac{f(x) + x_2 + sgn(\sigma)}{g(x)}$$

Example: $f(x) = x_1x_2,\ g(x) = 1,\ \sigma = x_1 + x_2$, yields

$$u = -x_1x_2 - x_2 - sgn(x_1 + x_2)$$

Step 3. The Equivalent Control

When trajectory reaches sliding mode, i.e., $x \in S$, then u will chatter (high frequency switching).

However, an equivalent control $u_{eq}(t)$ that keeps $x(t)$ on S can be computed from $\dot{\sigma} = 0$ when $\sigma = 0$

Example:

$$\dot{\sigma} = \dot{x}_1 + \dot{x}_2 = x_2 + x_1 x_2 + u_{eq} = 0 \quad \Rightarrow \quad u_{eq} = -x_2 - x_1 x_2$$

Thus, the sliding controller will take the system to the sliding manifold S in finite time, and the equivalent control will keep it on S.

Note!

Previous years it has often been assumed that the sliding mode control always is on the form

$$u = -sgn(\sigma)$$

This is OK, but is not completely general (see example)

Question 8

Can you repeat backstepping?

Backstepping Design

We are concerned with finding a stabilizing control $u(x)$ for the system

$$\dot{x} = f(x, u)$$

General Lyapunov control design: determine a Control Lyapunov function $V(x, u)$ and determine $u(x)$ so that

$$V(x) > 0 \,, \quad \dot{V}(x) < 0 \, \forall x \in \mathbb{R}^n$$

In this course we only consider $f(x, u)$ with a special structure, namely **strict feedback structure**

Strict Feedback Systems

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n) + g_n(x_1, \dots, x_n)u\end{aligned}$$

where $g_k \neq 0$

Note: $x_1, \dots, x_k$ do not depend on $x_{k+2}, \dots, x_n$.

The Backstepping Idea

Given a Control Lyapunov Function $V_1(x_1)$, with corresponding control $u = \phi_1(x_1)$, for the system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u$$

find a Control Lyapunov function $V_2(x_1, x_2)$, with corresponding control $u = \phi_2(x_1, x_2)$, for the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + u\end{aligned}$$

The Backstepping Result

Let $V_1(x_1)$ be a Control Lyapunov Function for the system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u$$

with corresponding controller $u = \phi(x_1)$.

Then $V_2(x_1, x_2) = V_1(x_1) + (x_2 - \phi(x_1))^2/2$ is a Control Lyapunov Function for the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + u\end{aligned}$$

with corresponding controller

$$u(x) = \frac{d\phi}{dx_1}\Big(f(x_1) + g(x_1)x_2\Big) - \frac{dV}{dx_1}g(x_1) - (x_k - \phi(x_1)) - f_2(x_1, x_2)$$

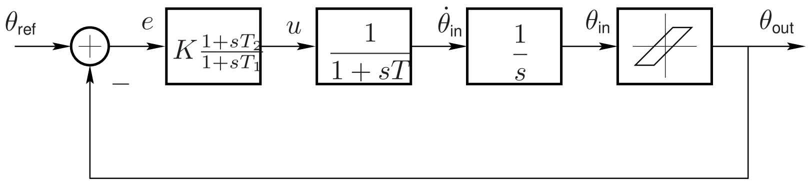
Question 9

Repeat backlash compensation

Backlash Compensation

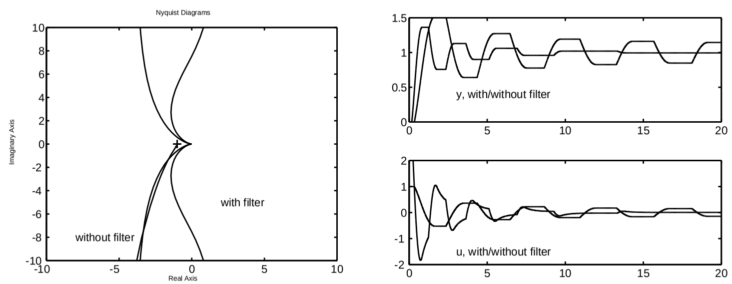
- Deadzone
- Linear controller design
- Backlash inverse

Linear controller design: Phase lead compensation



- Choose compensation $F(s)$ such that the intersection with the describing function is removed

$F(s) = K\frac{1+sT_2}{1+sT_1}$ with $T_1 = 0.5, T_2 = 2.0$:



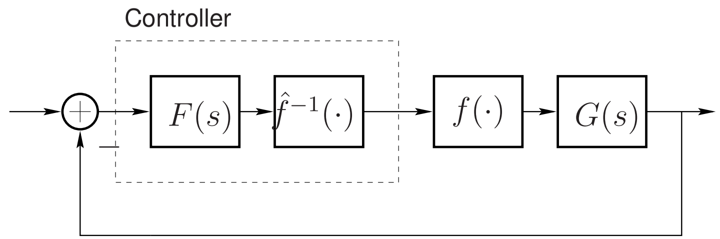
Oscillation removed!

Question 10

Can you repeat linearization through high gain feedback?

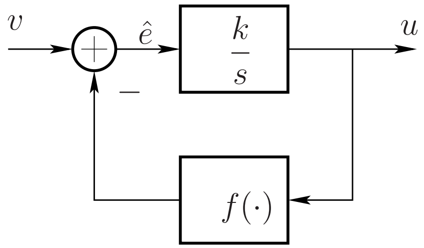
Inverting Nonlinearities

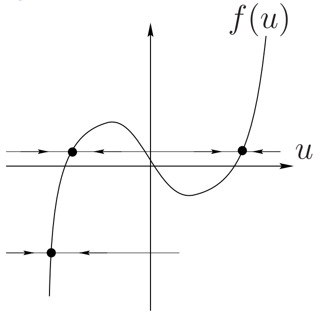
Compensation of static nonlinearity through inversion:



Should be combined with feedback as in the figure!

Remark: How to Obtain f^{-1} from f using Feedback





$$\hat{e} = \big(v - f(u)\big)$$

If $k > 0$ large and $df/du > 0$, then $\hat{e} \to 0$ and

$$0 = \big(v - f(u)\big) \quad \Leftrightarrow \quad f(u) = v \quad \Leftrightarrow \quad u = f^{-1}(v)$$

Question 11

What should we know about input–output stability?

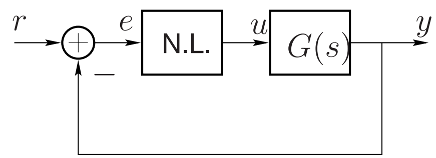
You should understand and be able to derive/apply

- System gain $\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2}$
- BIBO stability
- Small Gain Theorem
- Circle Criterion
- Passivity Theorem

Question 12

What about describing functions?

Idea Behind Describing Function Method



$e(t) = A \sin \omega t$ gives

$$u(t) = \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \sin[n\omega t + \arctan(a_n/b_n)]$$

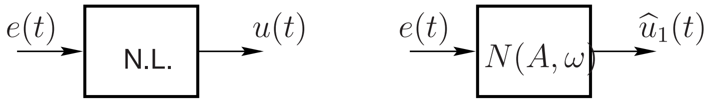
If $|G(in\omega)| \ll |G(i\omega)|$ for $n \geq 2$, then $n = 1$ suffices, so that

$$y(t) \approx |G(i\omega)| \sqrt{a_1^2 + b_1^2} \sin[\omega t + \arctan(a_1/b_1) + \arg G(i\omega)]$$

Definition of Describing Function

The **describing function** is

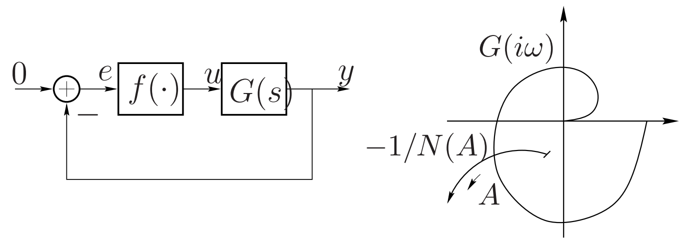
$$N(A,\omega) = \frac{b_1(\omega) + i a_1(\omega)}{A}$$



If G is low pass and $a_0 = 0$ then

$$\widehat{u}_1(t) = |N(A,\omega)| A \sin[\omega t + \arg N(A,\omega)] \approx u(t)$$

Existence of Periodic Solutions



$$y = G(i\omega)u = -G(i\omega)N(A)y \quad \Rightarrow \quad G(i\omega) = -\frac{1}{N(A)}$$

The intersections of the curves $G(i\omega)$ and $-1/N(A)$ give ω and A for a possible periodic solution.

More Courses in Control

- EL2450 Hybrid and Embedded Control Systems, per 3
- EL2745 Principles of Wireless Sensor Networks, per 3
- EL2520 Control Theory and Practice, Advanced Course, per 4
- EL1820 Modelling of Dynamic Systems, per 1
- EL2421 Project Course in Automatic Control, per 2

EL2520 Control Theory and Practice, Advanced Course

Aim: provide an introduction to principles and methods in advanced control, especially multivariable feedback systems.

- Period 4, 7.5 p
- Multivariable control:
  - Linear multivariable systems
  - Robustness and performance
  - Design of multivariable controllers: LQG, H_∞-optimization
  - Real time optimization: Model Predictive Control (MPC)
- Lectures, exercises, labs, computer exercises

Contact: Mikael Johansson `mikaelj@kth.se`

EL2450 Hybrid and Embedded Control Systems

Aim:course on analysis, design and implementation of control algorithms in networked and embedded systems.

- Period 3, 7.5 p
- How is control implemented in reality:
  - Computer-implementation of control algorithms
  - Scheduling of real-time software
  - Control over communication networks
- Lectures, exercises, homework, computer exercises

Contact: Dimos Dimarogonas `dimos@kth.se`

EL2745 Principles of Wireless Sensor Networks

Aim: provide the participants with a basic knowledge of wireless sensor networks (WSN)

- Period 3, 7.5 cr
- THE INTERNET OF THINGS
  - Essential tools within communication, control, optimization and signal processing needed to cope with WSN
  - Design of practical WSNs
  - Research topics in WSNs

Contact: Carlo Fischione `carlofi@kth.se`

EL1820 Modelling of Dynamic Systems

Aim: teach how to systematically build mathematical models of technical systems from physical laws and from measured signals.

- Period 1, 6 p
- Model dynamical systems from
  - physics: lagrangian mechanics, electrical circuits etc
  - experiments: parametric identification, frequency response
- Computer tools for modeling, identification, and simulation
- Lectures, exercises, labs, computer exercises

Contact: Håkan Hjalmarsson, `hjalmars@kth.se`

EL2421 Project Course in Control

Aim: provide practical knowledge about modeling, analysis, design, and implementation of control systems. Give some experience in project management and presentation.

- Period 4, 12 p
- “From start to goal...”: apply the theory from other courses
- Team work
- Preparation for Master thesis project
- Project management (lecturers from industry)
- No regular lectures or labs

Contact: Jonas Mårtensson `jonas1@kth.se`

Doing Master Thesis Project at Automatic Control Lab

- ○ Theory and practice
- ○ Cross-disciplinary
- ○ The research edge
- ○ Collaboration with leading industry and universities
- ○ Get insight in research and development

Hints:

- The topic and the results of your thesis are up to you
- Discuss with professors, lecturers, PhD and MS students
- Check old projects

Doing PhD Thesis Project at Automatic Control

- Intellectual stimuli
- Get paid for studying
- International collaborations and travel
- Competitive
- World-wide job market
- Research (50%), courses (30%), teaching (20%), fun (100%)
- 4-5 yr's to PhD (lic after 2-3 yr's)