FUNDAMENTAL PROPERTIES

ADVANCED TOPICS

- **Asymptotic covariance expressions (for high model orders)**
- **Proof of a.s. convergence results**

ASYMPTOTIC (IN MODEL ORDER) COVARIANCE EXPRESSIONS

Reminder: In SISO open loop, $S \in \mathcal{M}$, with G_θ and H_θ independently parameterized

$$\operatorname{cov} \hat{G}_N(e^{j\omega}) \approx \frac{1}{N} \Gamma^H(e^{j\omega}) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma(e^{j\tau}) \Gamma^H(e^{j\tau}) \frac{\Phi_u(\tau)}{\Phi_v(\tau)} d\tau \right]^{-1} \Gamma(e^{j\omega})$$

where
$$\Gamma(e^{j\omega}) = \left. \frac{\partial G_\theta(e^{j\omega})}{\partial \theta} \right|_{\theta=\theta_0}$$

Example: For FIR models $G_\theta(q) = b_1 q^{-1} + \cdots + b_n q^{-n}$, with $\theta = [\begin{array}{ccc} b_1 & \cdots & b_n \end{array}]^T$

$$\Gamma(z) = \begin{bmatrix} z^{-1} \\ \vdots \\ z^{-n} \end{bmatrix} \quad \Rightarrow \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma(e^{j\tau}) \Gamma^H(e^{j\tau}) \frac{\Phi_u(\tau)}{\Phi_v(\tau)} d\tau = \begin{bmatrix} r_0 & \cdots & r_{n-1} \\ \vdots & \ddots & \vdots \\ r_{n-1} & \cdots & r_0 \end{bmatrix} \quad \text{Toeplitz!}$$

ASYMPTOTIC (IN MODEL ORDER) COVARIANCE EXPRESSIONS (CONT.)

Let's continue with the FIR case: If $\Phi_u(\omega)/\Phi_v(\omega) = 1/\left|A(e^{j\omega})\right|^2$ (AR spectrum)

$$\tilde{\Gamma}(z) = \frac{1}{A(z)}\Gamma(z) \;\Rightarrow\; \operatorname{cov}\hat{G}_N(e^{j\omega}) \approx \frac{\left|A(e^{j\omega})\right|^2}{N}\tilde{\Gamma}^H(e^{j\omega})\underbrace{\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\Gamma}(e^{j\tau})\tilde{\Gamma}^H(e^{j\tau})\,d\tau\right]}_{M}^{-1}\tilde{\Gamma}(e^{j\omega})$$

Furthermore, this expression is invariant w.r.t. $\tilde{\Gamma}(e^{j\omega}) \to T\tilde{\Gamma}(e^{j\omega})$ (T nonsingular)
Hence, $\operatorname{cov}\hat{G}_N(e^{j\omega})$ only depends on the space spanned by the rows of $\tilde{\Gamma}(e^{j\omega})$

Idea: Choose T to make $M = I$!

I.e. Find an orthogonal basis for

$$\text{rowspace}\{\tilde{\Gamma}(z)\} = \left\langle \frac{z^{-1}}{A(z)},\cdots,\frac{z^{-n}}{A(z)} \right\rangle$$

ASYMPTOTIC (IN MODEL ORDER) COVARIANCE EXPRESSIONS (CONT.)

If $A(z) = (1 - p_1 z^{-1}) \cdots (1 - p_m z^{-m})$, with $m \leq n$, then by the Gram-Schmidt method,

$$\mathcal{B}_k(z) = \frac{\sqrt{1 - |p_k|^2}}{z - p_k} \prod_{l=1}^{k-1} \frac{1 - \bar{p}_l z}{z - p_l}, \qquad k = 1, \ldots, n \qquad (p_{m+1} = \cdots = p_n = 0)$$

are orthogonal functions which span rowspace$\{\tilde{\Gamma}(e^{j\omega})\}$. Then

$$\operatorname{cov} \hat{G}_N(e^{j\omega}) \approx \frac{|A(e^{j\omega})|^2}{N} [\ \overline{\mathcal{B}_1(e^{j\omega})} \ \cdots \ \overline{\mathcal{B}_n(e^{j\omega})}\ ] \begin{bmatrix} \mathcal{B}_1(e^{j\omega}) \\ \vdots \\ \mathcal{B}_n(e^{j\omega}) \end{bmatrix}$$

$$= \frac{1}{N} \frac{\Phi_v(\omega)}{\Phi_u(\omega)} \sum_{k=1}^{n} |\mathcal{B}_k(e^{j\omega})|^2$$

$$= \frac{1}{N} \frac{\Phi_v(\omega)}{\Phi_u(\omega)} \left[\sum_{k=1}^{m} \frac{1 - |p_k|^2}{|e^{j\omega} - p_k|^2} + n - m \right]$$

ASYMPTOTIC (IN MODEL ORDER) COVARIANCE EXPRESSIONS (CONT.)

When the model order n is large enough (but m, the order of $A(q)$, is fixed),

$$\operatorname{cov} \hat{G}_N(e^{j\omega}) \approx \frac{n}{N} \frac{\Phi_v(\omega)}{\Phi_u(\omega)}$$

Since most smooth spectra Φ_u / Φ_v can be uniformly approximated by an AR spectrum, this formula holds under very general conditions, and for most standard model structures (but it requires $\Phi_u(\omega) / \Phi_v(\omega)$ to be continuous and strictly above zero for all ω)

PROOF OF A.S. CONVERGENCE RESULTS

How to prove a.s. convergence?

- Method of subsequences

- Maximal inequalities

- Ergodic theorems

- Martingale theory

- "ODE" method

  ⋮

PROOF OF A.S. CONVERGENCE RESULTS (CONT.)

Borel-Cantelli Lemma: Let $\{A_k\}$ be a collection of events. Then

$$\sum_{k=1}^{\infty} P\{A_k\} < \infty \qquad \Rightarrow \qquad P\{\text{an infinite number of } A_k\text{'s ocur}\} = 0$$

Proof:

$$\begin{aligned}
P\{\text{infinite number of } A_k\text{'s ocur}\} &= P\{\text{for every N} \in \mathbb{N}, \text{ there is } k \geq N \text{ such that } A_k \text{ ocurs}\} \\
&= P\left\{\bigcap_{N=1}^{\infty} \{\text{there is } k \geq N \text{ such that } A_k \text{ ocurs}\}\right\} \\
&= P\left\{\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} A_k\right\} \\
&= \lim_{N\to\infty} P\left\{\bigcup_{k=N}^{\infty} A_k\right\} \\
&\leq \lim_{N\to\infty} \sum_{k=N}^{\infty} P\{A_k\} = \sum_{k=1}^{\infty} P\{A_k\} - \lim_{N\to\infty} \sum_{k=1}^{N-1} P\{A_k\} = 0
\end{aligned}$$

PROOF OF A.S. CONVERGENCE RESULTS (CONT.)

Chebyshev's Inequality: Let x be a random variable with d.f. F, and $\varepsilon > 0$. Then

$$P\{|x| > \varepsilon\} \le \frac{E\{x^2\}}{\varepsilon^2}$$

Proof:

$$E\{x^2\} = \int_{-\infty}^{\infty} x^2\, dF(x) \ge \int_{|x|>\varepsilon} x^2\, dF(x) \ge \varepsilon^2 \int_{|x|>\varepsilon} dF(x) = \varepsilon^2 P\{|x| > \varepsilon\}$$

PROOF OF A.S. CONVERGENCE RESULTS (CONT.)

Idea of a convergence proof: For fixed $\varepsilon > 0$, let $A_k = \{|x_k| > \varepsilon\}$. Then

$$\sum_{k=1}^{\infty} P\{A_k\} < \infty \quad \Rightarrow \quad P\{|x_k| > \varepsilon \quad \text{for finite number of } k\text{'s}\} = 1$$

$$\Rightarrow \quad P\{\text{there is } N \in \mathbb{N} \text{ such that for all } k \geq N,\ |x_k| > \varepsilon\} = 1$$

$$\Rightarrow \quad P\{\lim_{k\to\infty} x_k = 0\} = 1$$

$$\Rightarrow \quad x_k \xrightarrow[N\to\infty]{a.s.} 0$$

We can bound $P\{A_k\}$ by Chebyshev's inequality!

Sometimes Chebyshev's inequality is not strong enough, so we can only prove convergence for a *subsequence* $\{x_{n_k}\}$, but then we can use this to show that

$$\max_{n_k \leq i < n_{k+1}} |x_i| \xrightarrow[k\to\infty]{a.s.} 0$$

PROOF OF A.S. CONVERGENCE RESULTS (CONT.)

Thm (2B.1): Let

- $\{G_\theta(q): \theta \in D_\theta\}$, $\{H_\theta(q,t): \theta \in D_\theta, t \in \mathbb{N}\}$, $\{M_\theta(q): \theta \in D_\theta\}$ uniformly stable
- $\{w_t\}$ quasi-stationary and bounded by C_w
- $v_t = H_\theta(q,t)e_t$, where $\{e_t\}$ are independent random variables with $E\{e_t\} = 0$, $E\{e_t e_t^T\} = \Lambda_t$ and bounded moments of order 4
- $s_t(\theta) = G_\theta(q)v_t + M_\theta(q)w_t$

Then

$$\sup_{\theta \in D_\theta} \left\| \frac{1}{N} \sum_{t=1}^{N} [s_t(\theta)s_t^T(\theta) - E\{s_t(\theta)s_t^T(\theta)\}] \right\| \xrightarrow[N\to\infty]{a.s.} 0$$

Remark: This shows that $\sup_{\theta \in D_{\mathcal{M}}} \left\| V_N(\theta) - \bar{V}(\theta) \right\| \xrightarrow[N\to\infty]{a.s.} 0$, thus establishing the a.s. convergence of PEM

Proof of a.s. convergence results (cont.)

Proof of Thm (2B.1): Let $r_t(\theta) := s_t(\theta)s_t^T(\theta) - E\{s_t(\theta)s_t^T(\theta)\}$, and

$$R_r^N := \sup_{\theta \in D_\theta} \left\| \sum_{t=r}^{N} r_t(\theta) \right\|$$

We want to prove that $(1/N)R_1^N \xrightarrow[N\to\infty]{a.s} 0$.

After long calculations (see Lemma 2B.2), we have, for some $C > 0$,

$$E\{(R_r^N)^2\} \le C(N - r)$$

Therefore

$$E\left\{\frac{1}{N^2}(R_1^N)^2\right\} \le \frac{1}{N^2}C(N-1) = O\left(\frac{1}{N}\right) \qquad \text{and} \qquad \sum_{N=1}^{\infty} 1/N = \infty$$

$\Rightarrow$ Chebyshev+Borel-Cantelli cannot be directly used

PROOF OF A.S. CONVERGENCE RESULTS (CONT.)

However, $$E\left\{\left(\frac{1}{N^2}R_1^{N^2}\right)^2\right\} \le O\left(\frac{1}{N^2}\right) \quad \Rightarrow \quad (1/N^2)R_1^{N^2} \xrightarrow[N\to\infty]{a.s} 0$$

Let's study: $$\max_{N^2 \le k < (N+1)^2} \frac{1}{k} R_1^k$$

Then, if $k_N \in [N^2,(N+1)^2)$ and $\theta_N \in D_\theta$ maximize this quantity, we have

$$\begin{aligned}
\max_{N^2 \le k < (N+1)^2} \frac{1}{k} R_1^k &= \frac{1}{k_N}\left\|\sum_{t=1}^{k_N} r_t(\theta_N)\right\| \\
&\le \frac{1}{k_N}\left\|\sum_{t=1}^{N^2} r_t(\theta_N)\right\| + \frac{1}{k_N}\left\|\sum_{t=N^2+1}^{k_N} r_t(\theta_N)\right\| \\
&\le \underbrace{\frac{1}{N^2}R_1^{N^2}}_{\xrightarrow[N\to\infty]{a.s.} 0} + \frac{1}{N^2}R_{N^2+1}^{k_N}
\end{aligned}$$

PROOF OF A.S. CONVERGENCE RESULTS (CONT.)

Also,

$$E\left\{\left(\frac{1}{N^2}R_{N^2+1}^{k_N}\right)^2\right\} \leq \frac{1}{N^4}E\left\{\max_{N^2\leq k<(N+1)^2}\left(R_{N^2+1}^{k}\right)^2\right\}$$

$$\leq \frac{1}{N^4}\sum_{k=N^2}^{(N+1)^2-1}E\left\{\left(R_{N^2+1}^{k}\right)^2\right\}$$

$$\leq \frac{1}{N^4}\sum_{k=N^2}^{(N+1)^2-1}C(k-N^2-1)$$

$$\leq \frac{C'}{N^4}N^2 = \frac{C'}{N^2}$$

Hence, by Chebyshev+Borel-Cantelli, $(1/N^2)R_{N^2+1}^{k_N} \xrightarrow[N\to\infty]{a.s.} 0$, and so also $\max_{N^2\leq k<(N+1)^2}(1/k)R_1^k \xrightarrow[N\to\infty]{a.s.} 0$