COMPLEX NUMBERS

FUNDAMENTAL THEOREM OF ALGEBRA

* EVERY NON-ZERO, SINGLE VARIABLE, N DEGREE POLYNOMIAL HAS EXACTLY N ROOTS

$(x+1)^2 = -9$ (NO REAL SOLUTION)

$x+1 = \pm\sqrt{-9}$

$x+1 = \pm 3\sqrt{-1}$

$x = -1 \pm 3\sqrt{-1}$

$$\boxed{j^2 = -1 \text{ - IMAGINARY UNIT}}$$

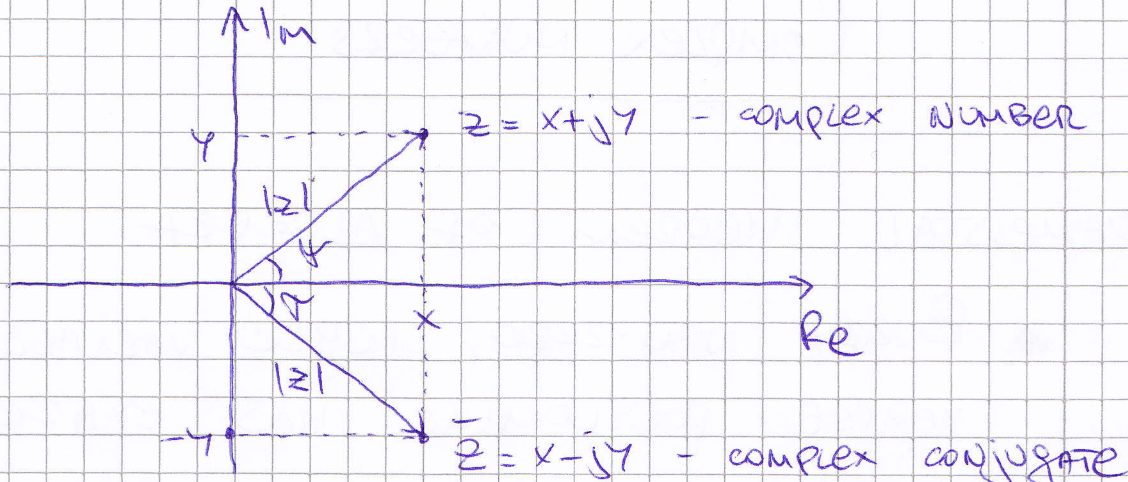
$x = -1 \pm 3j$

$$\boxed{z = x + jy \text{ - COMPLEX NUMBER}}$$

$Re\{z\} = x$ - REAL PART

$Im\{z\} = y$ - IMAGINARY PART

$$\boxed{\text{CARDANO - }16^{TH}\text{ CENTURY}}$$



$x = |z| \cdot \cos\vartheta$

$y = |z| \cdot \sin\vartheta$

$|z| = \sqrt{x^2 + y^2}$

$$\boxed{\begin{array}{l} z = x + jy \text{ - cartesian form} \\ z = |z|(\cos\vartheta + j \cdot \sin\vartheta) \text{ - complex form} \end{array}}$$

Euler's formula

* For any real number **x**,

$$\boxed{e^{jx} = \cos x + j \cdot \sin x}$$

where **e** is the base of the natural logarithm

$$\boxed{\begin{array}{ll} z = |z| \cdot e^{j\vartheta} & \bar{z} = |z|\,e^{-j\vartheta} \\ z = x + jy & \bar{z} = x - jy \end{array}}$$

(1)

a) SHOW THAT $z.\bar{z} = |z|^2$

$z = x + jy$

$\bar{z} = x - jy$

$\Rightarrow z.\bar{z} = (x+jy)(x-jy) = x^2 + y^2 \quad (1)$

$|z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2 \quad (2)$

$\Rightarrow \underline{\underline{z.\bar{z} = |z|^2}}$

b) SHOW THAT $\overline{z+w} = \bar{z} + \bar{w}$

LET $z = a + jb$ AND $w = c + jd$

$\Rightarrow \overline{z+w} = \overline{(a+jb)+(c+jd)} = \overline{(a+c)+j(b+d)} =$

$= (a+c) - j(b+d) \quad (1)$

$\Rightarrow \bar{z} + \bar{w} = \overline{(a+jb)} + \overline{(c+jd)} = a - jb + c - jd =$

$= (a+c) - j(b+d)$

$\Rightarrow \underline{\underline{\overline{z+w} = \bar{z} + \bar{w}}}$

c) SHOW THAT $\overline{z.w} = \bar{z}.\bar{w}$

LET $z = a + jb$ AND $w = c + jd$

$$\Rightarrow \overline{z.w} = \overline{(a+jb)(c+jd)} = \overline{ac + ajd + cjb + \underbrace{j^2}_{=-1} bd} =$$

$$= \overline{\underbrace{(ac - bd)}_{Re} + j\underbrace{(ad + cb)}_{Im}} = (ac - bd) - j(ad + cb) \quad (1)$$

$$\Rightarrow \bar{z}.\bar{w} = (a - jb).(c - jd) = ac - ajd - cjb + \underbrace{j^2}_{=-1} bd =$$

$$= (ac - bd) - j(ad + cb) \quad (2)$$

$$\Rightarrow \overline{z.w} = \bar{z}.\bar{w}$$

(2) WRITE EACH OF THE FOLLOWING IN THE FORM $a + jb$

a) $e^{-\frac{2\pi}{3}j}$

$$= \cos\left(-\frac{2\pi}{3}\right) + j.\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

b) $12e^{\frac{\pi}{6}j}$

$$= 12\left(\cos\left(\frac{\pi}{6}\right) + j.\sin\left(\frac{\pi}{6}\right)\right) = 12\left(\sqrt{3}/2 + j.1/2\right) = 6\sqrt{3} + 6j$$

c) $\underbrace{e^{\frac{2\pi}{3}j}}_{1} + \underbrace{e^{\frac{4\pi}{3}j}}_{2} + \underbrace{e^{\frac{6\pi}{3}j}}_{3}$

$1 \Rightarrow \cos\left(\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + j\sqrt{3}/2$

$2 \Rightarrow \quad \cos\left(\frac{4\pi}{3}\right) + j\sin\left(\frac{4\pi}{2}\right) = -\frac{1}{\underline{2}} - j\sqrt{3}/2$

$3 \Rightarrow \quad \cos(2\pi) + j\sin(2\pi) = 1 + 0 \cdot j = 1$

$\Rightarrow -\frac{1}{2} + \cancel{j\frac{\sqrt{3}}{2}} - \frac{1}{2} - \cancel{j\frac{\sqrt{3}}{2}} + 1 = 0$ //

③ SHOW THAT $\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$ AND

$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$, FOR ALL REAL ϕ.

$$\frac{e^{j\phi} + e^{-j\phi}}{2} = \frac{\cos(\phi) + j\sin(\phi) + \cos(\phi) - j\sin(\phi)}{2} = \frac{\cancel{2}\cos(\phi)}{\cancel{2}} //$$

$$\frac{e^{j\phi} - e^{-j\phi}}{2j} = \frac{\cos(\phi) + j\sin(\phi) - (\cos(\phi) - j\sin(\phi))}{2j} =$$

$$= \frac{\cancel{\cos(\phi)} + j\sin(\phi) - \cancel{\cos(\phi)} + j\sin(\phi)}{2j} = \frac{\cancel{2j}\sin(\phi)}{\cancel{2j}} //$$

④ SHOW THAT $(-1+j)^7 = -8(1+j)$

POLAR FORM OF $(-1+j)$

$|z| = \sqrt{(\underset{x}{-1})^2 + \underset{y}{1}^2} = \sqrt{2}$

$$\left.\begin{array}{l} \cos\theta = \frac{x}{|z|} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \\ \sin\theta = \frac{y}{|z|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{array}\right\} \phi = \frac{3\pi}{4} \quad (135^\circ)$$

$$\Rightarrow (-1+j) = \sqrt{2}\, e^{j\frac{3\pi}{4}}$$

$$\Rightarrow (-1+j)^7 = \left(\sqrt{2}\, e^{j\frac{3\pi}{4}}\right)^7 = \sqrt{2}^7 e^{j\frac{21\pi}{4}} = \sqrt{2}^7 \cdot e^{j4\pi + j\frac{5\pi}{4}} =$$

$$= \sqrt{2}^7 \cdot e^{j4\pi} \cdot e^{j\frac{5\pi}{4}}$$

$$e^{j4\pi} = \underbrace{\cos(4\pi)}_{=1} + j\underbrace{\sin(4\pi)}_{=0} = 1$$

$$\Rightarrow \sqrt{2}^7 \cdot e^{j\frac{5\pi}{4}} = \sqrt{2}^7 \left(\underbrace{\cos\left(\frac{5\pi}{4}\right)}_{=-\frac{\sqrt{2}}{2}} + j\underbrace{\sin\left(\frac{5\pi}{4}\right)}_{=-\frac{\sqrt{2}}{2}} \right) =$$

$$= \sqrt{2}^7 \left(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \right) = \sqrt{2}^7 \left(-\frac{\sqrt{2}}{2}(1+j) \right) = -\frac{\sqrt{2}^8}{2}(1+j) =$$

$$= -\frac{16}{2}(1+j) = -8(1+j)$$