Linear Operators and Fourier Transform

DD2423 Image Analysis and Computer Vision

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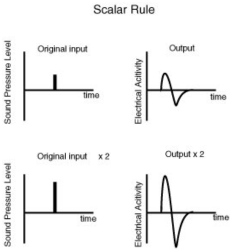
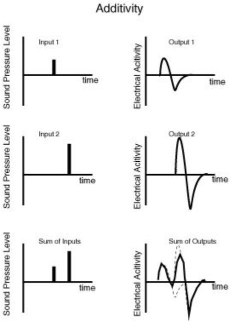
Linear Systems

Image processing operations can be modeled by utilizing linear systems theory. A linear system obeys the principle of superposition:

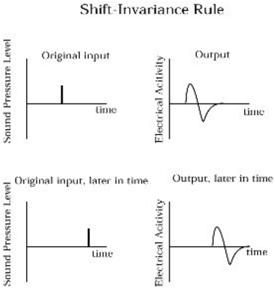
- Homogeneity (scalar rule): an increase in strength of the input, increases the output/response for the same amount.
- Additivity: if the input consists of two signals, the output/response is equal to the sum of the individual responses.

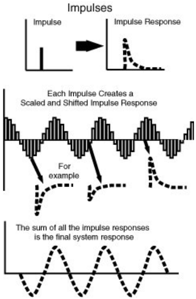




Additional properties:

- Shift-invariance: If a system is given two impulses with a time delay, the response remains the same except for time difference.
- Signals can be represented as sums of impulses of different strenghts (image intensities), shifted in time (image space).
- If we know how system responds to an impulse, we know how it reacts to combination of impulses: **impulse-response function**.





Notation

- Assume f and f' are 2D images, then $f \overset{\mathcal{L}}{\Longrightarrow} f' = \mathcal{L}(f)$,
  where $\mathcal{L}$ is an operator that "converts" the input f into the output f'.

Linear operator $\mathcal{L}$ satisfies

- Homogeneity: $\mathcal{L}(\alpha f(x,y)) = \alpha\mathcal{L}(f(x,y))$; $\alpha \in \mathbb{R}$
- Additivity: $\mathcal{L}(f(x,y)+g(x,y)) = \mathcal{L}(f(x,y)) + \mathcal{L}(g(x,y))$; $x, y \in \mathbb{R}$

Given

- $g \rightarrow \boxed{\quad\mathcal{L}\quad} \rightarrow \mathcal{L}(g)$
- $f \rightarrow \boxed{\quad\mathcal{L}\quad} \rightarrow \mathcal{L}(f)$

we have

- $(\alpha f + \beta g) \rightarrow \boxed{\quad\mathcal{L}\quad} \rightarrow \alpha\mathcal{L}(f) + \beta\mathcal{L}(g)$

Linear Shift Invariant Systems

$\mathcal{L}$ is called shift-invariant, if and only if a shift (translation) of the input causes the same shift of the output:

$$f(x,y) \rightarrow \boxed{\quad \mathcal{L} \quad} \rightarrow \mathcal{L}(f(x,y))$$

$$f(x-x_0, y-y_0) \rightarrow \boxed{\quad \mathcal{L} \quad} \rightarrow \mathcal{L}(f(x-x_0, y-y_0))$$

Alternative formulation: $\mathcal{L}$ commutes with a shift operator $\mathcal{S}$

$$\rightarrow \boxed{\quad \mathcal{L} \quad} \rightarrow \boxed{\quad \mathcal{S} \quad} \rightarrow \text{ same as } \rightarrow \boxed{\quad \mathcal{S} \quad} \rightarrow \boxed{\quad \mathcal{L} \quad} \rightarrow$$

$$\begin{tikzcd}
\mathbf{S(f)} \arrow[r, "\boldsymbol{L}"] & \mathbf{same!} \\
\mathbf{f} \arrow[u, "\mathbf{S}"] \arrow[r, "\boldsymbol{L}"'] & \boldsymbol{L}\mathbf{(f)} \arrow[u, "\mathbf{S}"']
\end{tikzcd}$$

Linear filtering in image processing

Using digital linear filters to modify pixel values based on some pixel neighborhoods. Linear means linear combination of neighbors.

- Linear methods simplest.
- Can combine linear methods in any order to achieve same result.
- May be easier to invert.

Useful to:

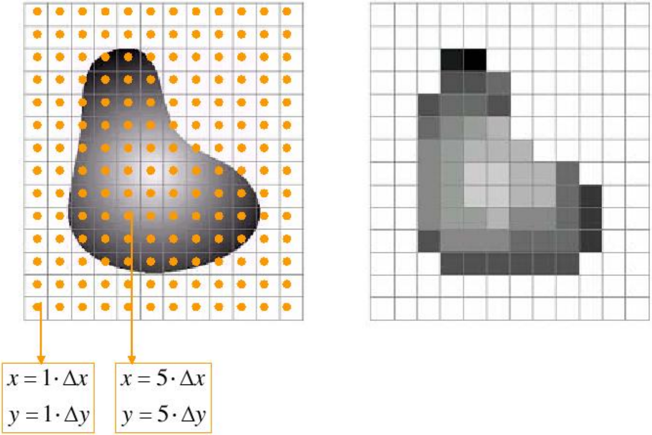
- Integrate information over constant regions.
- Scale changes of image intensities.
- Detect changes (edge detection).

Sampling

Our signals (images) are not in a continuous domain, but in a discrete.

- A continuous function $f(x, y)$ (an image) can be sampled using a discrete grid of sampling points.
- The image is sampled at points $(j\Delta x, k\Delta y)$, with $j = 1, ..., M$ and $k = 1, ..., N$, where is (M, N) is the size of the image in pixels.
- Here Δx and Δy are called the sampling interval.

Sampling



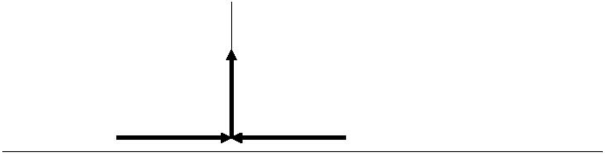
Sampling

Dirac (continuous domain) and Kronecker (discrete) delta functions.

Ideal impulse defined using ***Dirac distribution***

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)\,dx\,dy = 1$$

and $\delta(x,y) = 0$ *for all* $x, y \neq 0$



Sampling

The 'sifting property' of the dirac distribution provides a value of the function f(x,y) at point a,b

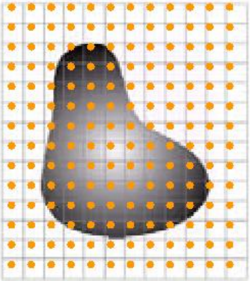
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-a,y-b)dxdy = f(a,b)$$

The sifting equation can be used to describe the sampling process of a continuous image function f(x,y)

Sampling

The ideal sampling s(x,y) in the regular grid can be represented using a collection of Dirac distributions δ

$$s(x,y) = \sum_{j=1}^{M} \sum_{k=1}^{N} \delta(x - j\Delta x, y - k\Delta y)$$

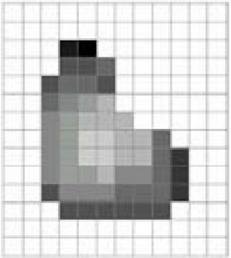


Sampling

The sampled image $f_s(x,y)$ is the product of the continuous image $f(x,y)$ and the sampling function $s(x,y)$

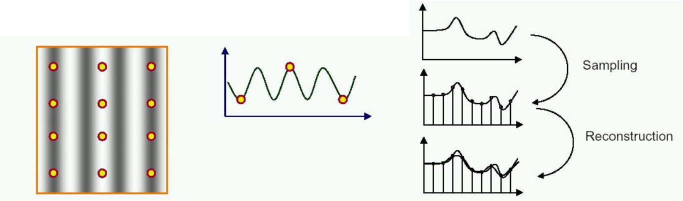
$$f_s(x,y) = f(x,y)s(x,y)$$

$$= f(x,y)\sum_{j=1}^{M}\sum_{k=1}^{N}\delta(x - j\Delta x, y - k\Delta y)$$



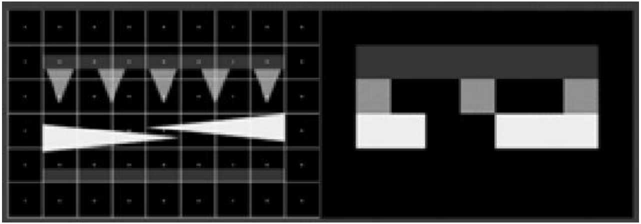
Sampling

- Sources of error during sampling:
  - Intensity quantization (not enough intensity resolution).
  - Spatial aliasing (not enough spatial resolution).
  - Temporal aliasing (not enough temporal resolution).
- Sampling Theorem answers (more later):
  - How many samples are required to describe the given signal without loss of information? (more later)
  - What signal can be reconstructed given the current sampling rate?

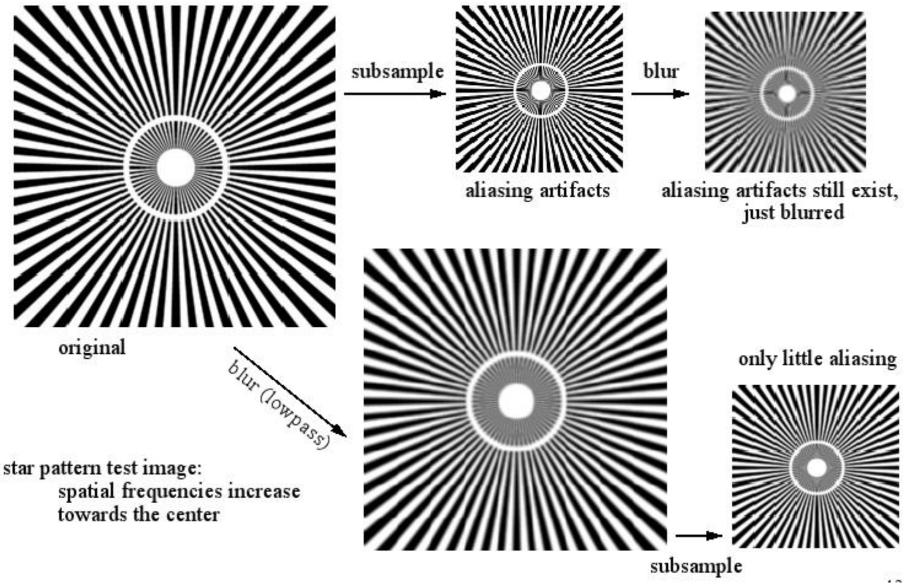


Aliasing and anti-aliasing

- Artifacts produced by under-sampling or poor reconstruction.
  Fine structures disappear and distort coarser structure.
- Spatial and temporal aliasing.
- Anti-aliasing: sample at higher rate or prefiltering.
  Tools: Fourier transform, convolution and sampling theory.

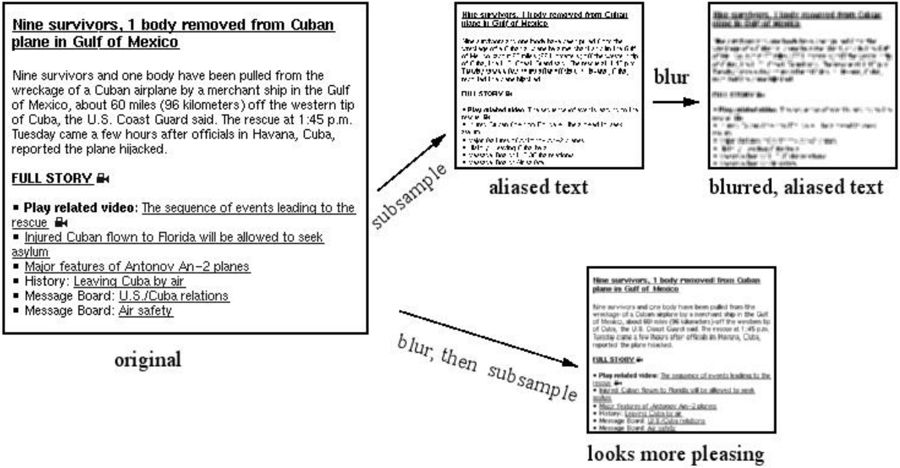


Example: Aliasing



Example: Aliasing

Low pass filtering (bluring) important!



Linear image filtering

- Estimate an output image by modifying pixels in the input image using a function of a local pixel neighborhood.
- The neighborhood and the corresponding linear weights per pixel is called a convolution kernel.

$$\begin{array}{|c|c|c|}\hline 9 & 5 & 3 \\ \hline 4 & 5 & 1 \\ \hline 1 & 1 & 7 \\ \hline \end{array} \quad \overset{\text{some function}}{\Longrightarrow} \quad \begin{array}{|c|c|c|}\hline & & \\ \hline & 9 & \\ \hline & & \\ \hline \end{array}$$

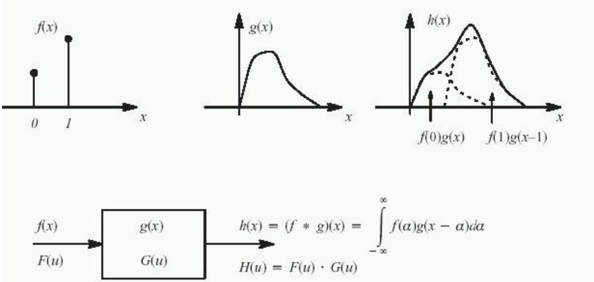
$$\begin{array}{|c|c|c|}\hline 9 & 5 & 3 \\ \hline 4 & 5 & 1 \\ \hline 1 & 1 & 7 \\ \hline \end{array} * \begin{array}{|c|c|c|}\hline 0 & -1 & 0 \\ \hline -1 & 4 & -1 \\ \hline 0 & -1 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline & & \\ \hline & 9 & \\ \hline & & \\ \hline \end{array}$$

Convolution

- Convolution is a tool to build linear shift invariant (LSI) filters.
- Mathematically, a convolution is defined as the integral over space of one function at α, times another function at $x-\alpha$.

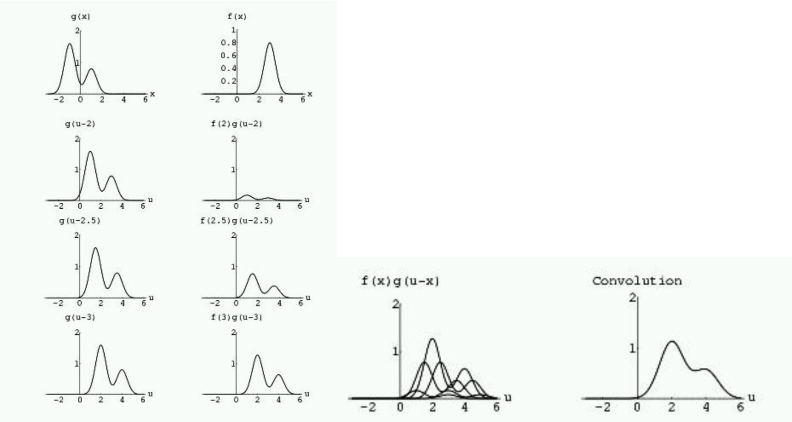
$$f(x) * g(x) = \int_{\alpha \in \mathbb{R}^n} f(\alpha) g(x-\alpha) d\alpha = g(x) * f(x) = \int_{\alpha \in \mathbb{R}^n} g(\alpha) f(x-\alpha) d\alpha$$

- Convolution operation is commutative!



Convolutions as weighted sums

Way of considering convolution: weighted sum of shifted copies of one function, with weights given by the function value of the second function at the shift vector.



Theorem 1

Every shift invariant linear operator can be written as a convolution

$$\mathcal{L}(\mathrm{f}) = \mathrm{g} * \mathrm{f}$$

- Continuous case

$$\mathcal{L}(\mathrm{f}(x)) = \int_{\alpha \in \mathbb{R}^n} \mathrm{g}(\alpha)\mathrm{f}(x-\alpha)d\alpha$$

- Discrete case

$$\mathcal{L}(\mathrm{f}(x)) = \sum_{\alpha \in \mathbb{R}^n} \mathrm{g}(\alpha)\mathrm{f}(x-\alpha)$$

Convolution (discrete case)

- The convolution of an image $f(x,y)$ with a kernel h(x,y) is

$$g(x,y) = h(x,y) * f(x,y) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} h(m,n)\, f(x-m, y-n)$$

- Convolution kernel $h(x,y)$ represented as a matrix and is also called:
  - impulse response,
  - point spread function,
  - filter kernel,
  - filter mask,
  - template...

Convolution (filtering)

- Frame mask over image - multiply mask values by image values and sum up the results - a sliding dot product.

- For mathematical correctness: From the definition, the kernel first has to be flipped x-wise and y-wise. People are sloppy though.

Convolution: 1D example

If

$$F_1 = [1 \quad 2 \quad 3 \quad 4 \quad 5]$$
$$F_2 = [1 \quad 2 \quad 1 \quad 2 \quad 1]$$
$$G_1 = [-1 \quad 2 \quad -1]$$
$$G_2 = [1 \quad 2 \quad 3]$$

then

$$F_1 * G_1 = [-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 6 \quad -5]$$
$$F_2 * G_1 = [-1 \quad 0 \quad 2 \quad -2 \quad 2 \quad 0 \quad -1]$$
$$F_1 * G_2 = [1 \quad 4 \quad 10 \quad 16 \quad 22 \quad 22 \quad 15]$$
$$F_2 * G_2 = [1 \quad 4 \quad 8 \quad 10 \quad 8 \quad 8 \quad 3]$$

Note1: outside the windows, values are assumed to be zero.
Note2: normally you assume $x = 0$ at center of filter kernel.

Convolution: 1D example

$F_1 = [1 \quad 2 \quad 3 \quad 4 \quad 5]$

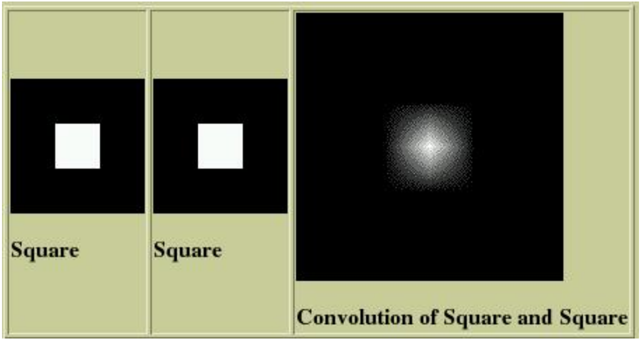
$G_2 = [1 \quad 2 \quad 3]$

$$
\begin{array}{cccccccc}
 & & & 1 & 2 & 3 & 4 & 5 \\
* & & & & & 1 & 2 & 3 \\
\hline
 & & & 3 & 6 & 9 & 12 & 15 \\
 & & 2 & 4 & 6 & 8 & 10 & \\
+ & 1 & 2 & 3 & 4 & 5 & & \\
\hline
 & 1 & 4 & 10 & 16 & 22 & 22 & 15
\end{array}
$$

An easier way of doing it! Almost like regular multiplication.

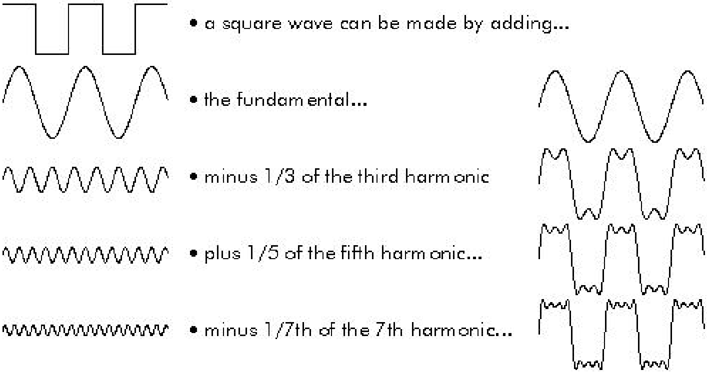
Convolution: 2D example

- Convolution of two images: since the squares have the same image and size, their convolution creates a gradient with the brightest spot in the center.



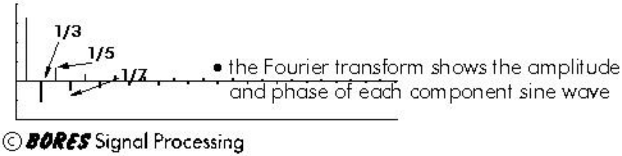
Signal decomposition

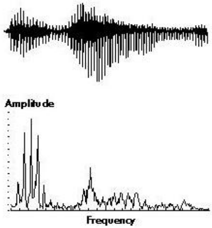
- In 1807 Jean Baptiste Fourier showed that any periodic signal could be represented by a series/sum of sine waves with appropriate amplitude, frequency and phase.



The Fourier transform

- The Fourier transform is an equation to calculate the frequency, amplitude and phase of each sine wave needed to make up any given signal.
- Using the Fourier transform we can study frequency components of signals.
- The Fourier transform *converts* between the spatial and frequency domain representations.



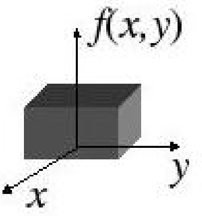


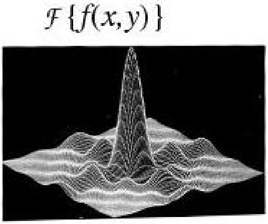
The Fourier transform

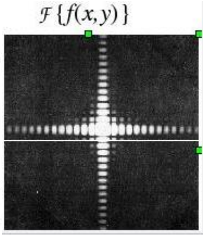
- An important image processing tool which is used to decompose an image into its sine and cosine components.
- The output of the transformation represents the image in the Fourier or frequency space, while the input image is the real space equivalent.
- In the Fourier space image, each point represents a particular frequency contained in the real domain image.
- The Fourier Transform is used in a wide range of applications, such as image analysis, image filtering, image reconstruction and image compression

Images and spatial frequency

- The spatial frequency of an image refers to the rate at which the pixel intensities change.
- The easiest way to determine the frequency composition of signals is to inspect that signal in the frequency domain.
- The frequency domain shows the magnitude of different frequency components.







Change of basis functions

- An image can be viewed as a spatial array of gray level values, but can also thought of as a spatially varying function.
- Decompose the image into a set of orthogonal basis functions.
- When basis functions are combined (linearly) the original function will be reconstructed.
- Spatial domain: basis consists of shifted Dirac functions.
  Fourier domain: basis consists of complex exponential functions.
- The Fourier transform is “just” a change of basis functions.

Change of basis functions

- Assume you have a vector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- This may be expressed with another basis (e.g. Haar wavelet).

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 2.5 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} - 0.5 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - 0.5 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

- The only condition is that the basis vector are orthogonal.

$$\mathcal{F}(\mathrm{f}(x)) = \int_{x\in\mathbb{R}^n} \mathrm{f}(x) e^{-i\omega^T x} dx = \hat{\mathrm{f}}(\omega)$$

$$\mathcal{F}^{-1}(\hat{\mathrm{f}}(\omega)) = \frac{1}{(2\pi)^n} \int_{\omega\in\mathbb{R}^n} \hat{\mathrm{f}}(\omega) e^{i\omega^T x} d\omega$$

$$e^{i\omega^T x} = \cos \omega^T x + i \sin \omega^T x$$

Terminology:

Frequency spectrum : $\hat{\mathrm{f}}(\omega) = R(\omega) + iI(\omega) = |\ \hat{\mathrm{f}}(\omega)\ |\ e^{i\phi(\omega)}$

Fourier spectrum: $|\ \hat{\mathrm{f}}(\omega)\ | = \sqrt{R^2(\omega) + I^2(\omega)}$

Power spectrum: $|\ \hat{\mathrm{f}}(\omega)\ |^2$

Phase angle: $\phi(\omega) = \arg \hat{\mathrm{f}}(\omega) = \tan^{-1} \frac{I(\omega)}{R(\omega)}$

Terminology

- Angular frequency $\omega = (\omega_1 \; \omega_2)^T$

$$\omega_1 = \text{angular frequency in } x \text{ direction}$$
$$\omega_2 = \text{angular frequency in } y \text{ direction}$$

- Frequency

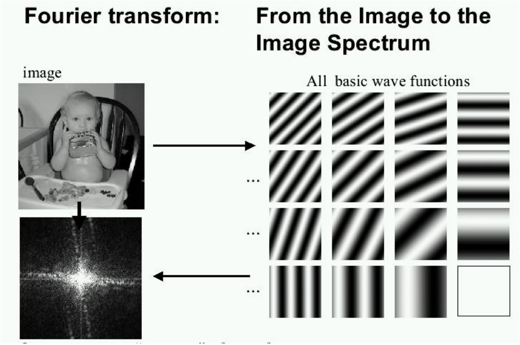
$$f = \frac{\omega}{2\pi}$$

- Wavelength

$$\lambda = \frac{2\pi}{\|\omega\|} = \frac{2\pi}{\sqrt{\omega_1^2 + \omega_2^2}}$$

$$e_\omega(x) = e^{i\omega^T x} = e^{i(\omega_1 x_1 + \omega_2 x_2)} = \cos\omega^T x + i\sin\omega^T x \quad \text{(Euler's formula)}$$

$$Re(e_\omega(x)) = \cos(\omega^T x) \ \text{ and } \ Im(e_\omega(x)) = \sin(\omega^T x), \ e_\omega : \mathbb{R}^2 \rightarrow \mathbb{C}$$



Magnitude and Phase

- The Fourier coefficients $F(u,v)$ are complex numbers, but it is not obvious what the real and imaginary parts represent.
- Another way to represent the data is with phase and magnitude.

  Magnitude:

  $$|F(u,v)| = \sqrt{Re^2(u,v) + Im^2(u,v)}$$

  Phase:

  $$\theta(u,v) = \tan^{-1}\frac{Im(u,v)}{Re(u,v)}$$

Note: Change in notation from $\hat{f}(\omega_1, \omega_2)$ to $F(u,v)$, which is more common, if we know we are working with images in two dimensions.

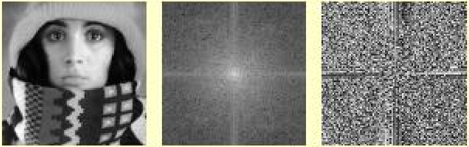


Figure 7 Illustration of an original image, magnitude and phase of the Fourier transform respectively

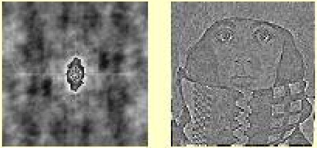
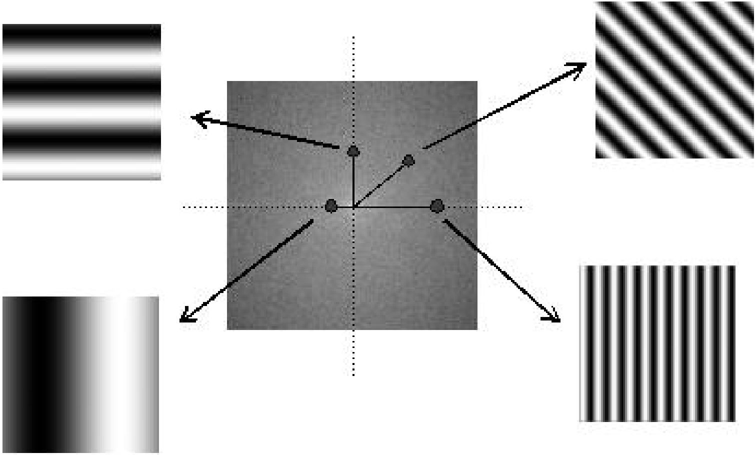
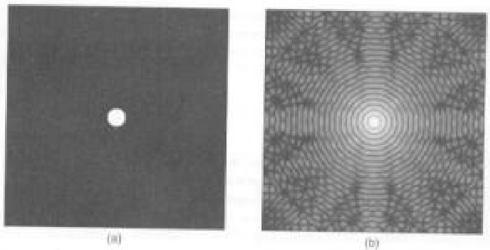


Figure 8 reconstructed images using only magnitude and phase

Example



- An image of a spot (left) and the Fourier Transform (right).



- In the frequency domain, u represents the spatial frequency along the original image's x axis and v represents the spatial frequency along the y axis. The origin is in the center of the image.

Summary of good questions

- What properties does a linear system have?
- What does shift-invarience mean in terms of image filtering?
- How can a Dirac function be used to model sampling?
- How do you define a convolution?
- Why are convolutions important in linear filtering?
- How do you define a 2D Fourier transform?
- If you apply a Fourier transform to an image, what do you get?
- What information does the phase contain? What about the magnitude?
- Will the second lab be as boring as the first?

Readings

- Gonzalez and Woods: Chapter 4
- Szeliski: Chapters 3.2 and 3.4
- Introduction to Lab 2