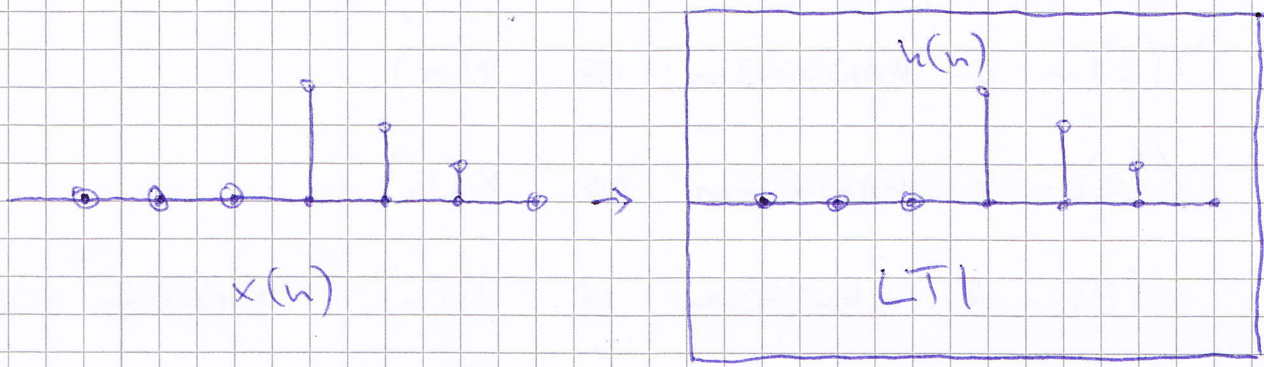
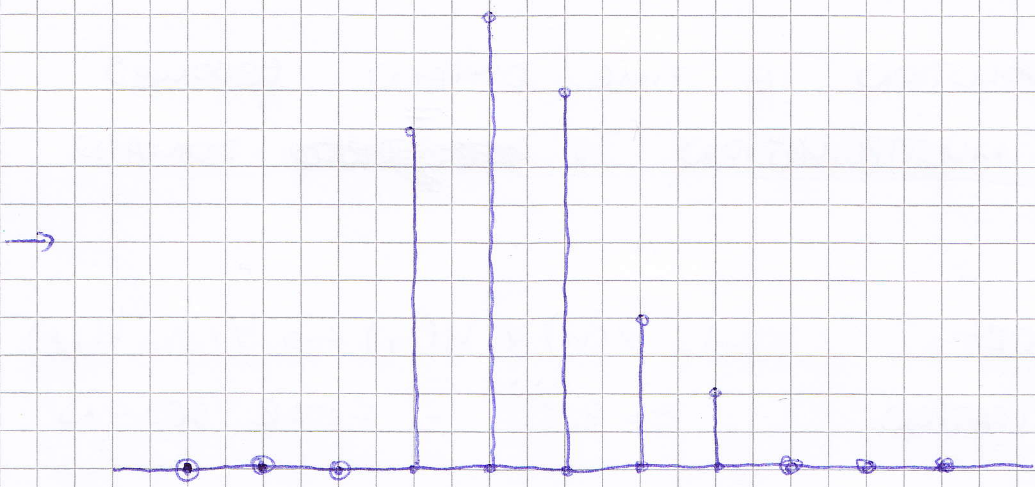
Z - TRANSFORM





$y(n) = x(n) * h(n)$

* CONVERTS DISCRETE-TIME DOMAIN SIGNAL INTO z DOMAIN REPRESENTATION

$$\boxed{X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}}$$

$z = A \cdot e^{j\theta} = A(\cos\theta + j \cdot \sin\theta)$ - COMPLEX NUM

A - MAGNITUDE OF z

j - IMAGINARY UNIT

θ - PHASE

$Y(z) = H(z) \cdot X(z)$

$Y(z)$ - Z-TRANSFORM OF $y(n)$

$X(z)$ - Z-TRANSFORM OF $x(n)$

$H(z)$ - Z-TRANSFORM OF $h(n)$ - TRANSFER FUNC

* CONVOLUTION IN TIME DOMAIN BECOMES MULTIPLICATION IN Z DOMAIN

* PROPERTY $y(n) = x(n) * h(n) \longleftrightarrow Y(z) = H(z) \cdot X(z)$

  IMPLICATION $z = e^{j\omega}$ - FILTER DESIGN

* PROPERTY $y(n) = y_1(n) + y_2(n) \longleftrightarrow Y(z) = Y_1(z) + Y_2(z)$

  IMPLICATION CONVERTING TO SUM
  - COMPLETING THE SQUARE
  - PARTIAL FRACTIONS

  USE OF TRANSFORM PAIRS

$x(n)$	$X(z)$	
$\delta(n)$	1	$\forall z$
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z \rvert > 1$

① FIND THE Z-TRANSFORM OF

$$x(n) = \delta(n)$$

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} \delta(n) \cdot z^{-n} \overset{n=0}{=\!=} 1 \cdot z^{-0} = 1, \quad \forall z$$

② FIND THE Z-TRANSFORM OF

$$x(n) = u(n)$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} u(n) \cdot z^{-n} \overset{n\geq 0}{=\!=} \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}$$

ASSUME THAT $|z^{-1}| < 1$ - converge

$$\Rightarrow S = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots \quad \left(\cdot \frac{1}{z}\right)$$

$$\frac{S}{z} = \frac{1}{z} + \frac{1}{z^2} + \ldots$$

$$\Rightarrow S - \frac{S}{z} = 1 \Rightarrow S\left(1 - \frac{1}{z}\right) = 1 \Rightarrow S = \frac{1}{1 - z^{-1}}, \; |z^{-1}| < 1$$

③ FIND THE Z-TRANSFORM OF

$$x(n) = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \cos\left(\frac{\pi n}{4}\right) \cdot u(n)$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

LET,

$$a = \frac{1}{\sqrt{2}} \quad \text{AND} \quad \omega = \frac{\pi}{4}$$

THEN,

$$x(n) = a^n \cdot \cos(\omega n) \cdot u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n \cdot \cos(\omega n) \cdot u(n) \cdot z^{-n}$$

$$X(z) \overset{n \geq 0}{=} \sum_{n=0}^{\infty} a^n \cdot \cos(\omega n) \cdot u(n) \cdot z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \frac{a^n}{z^n} \left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right)$$

$$X(z) = \frac{1}{2} \sum_{n=0}^{\infty} \left[\left(\frac{a \cdot e^{j\omega}}{z}\right)^n + \left(\frac{a \cdot e^{-j\omega}}{z}\right)^n\right]$$

$$X(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{a.e^{jw}}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{a.e^{-jw}}{z}\right)^n\right]$$

FROM ex. 2 $\boxed{\sum_{n=0}^{\infty}(z^{-1})^n = \frac{1}{1-z^{-1}}}$

$$\Rightarrow \quad X(z) = \frac{1}{2}\left[\frac{1}{1-\frac{a.e^{jw}}{z}} + \frac{1}{1-\frac{a.e^{-jw}}{z}}\right]$$

$$X(z) = \frac{1}{2}\left(\frac{z}{z-a.e^{jw}} + \frac{z}{z-a.e^{-jw}}\right)$$

$$X(z) = \frac{z}{2}\left(\frac{z-a.e^{-jw} + z-a.e^{jw}}{(z-a.e^{jw})(z-a.e^{-jw})}\right)$$

$$X(z) = \frac{z}{2}\left(\frac{2z - a\left(e^{-jw}+e^{jw}\right)}{z^2 - aze^{-jw} - a.z.e^{jw} + a^2}\right)$$

$$X(z) = \frac{z}{\cancel{2}}\left(\frac{\cancel{2}z - \cancel{2}a.\cos w}{z^2 - 2az.\cos w + a^2}\right)$$

$$X(z) = \frac{z^2 - a.z.\cos w}{z^2 - 2a.z.\cos w + a^2}$$

BUT,

$$\cos w = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$a = \frac{1}{\sqrt{2}}$$

$$\Rightarrow X(z) = \frac{z^2 - \frac{1}{\cancel{\sqrt{2}}} \cdot \frac{\cancel{\sqrt{2}}}{2} \cdot z}{z^2 - \frac{2}{\cancel{\sqrt{2}}} \cdot \frac{\cancel{\sqrt{2}}}{\cancel{2}} \cdot z + \frac{1}{2}} = \frac{z^2 - \frac{1}{2} z}{z^2 - z + \frac{1}{2}}$$

BUT, $|z^{-1}| < 1$, converge

$$\Rightarrow \left| \frac{a \cdot e^{jw}}{z} \right| < 1$$

$$|a \cdot e^{jw}| < |z|$$

$$|a| \cdot \underset{\overset{\shortparallel}{1}}{|e^{jw}|} < |z| \Rightarrow |z| > |a| \Rightarrow |z| > \frac{1}{\sqrt{2}}$$

Poles and Zeros

$$y(n) = 2y(n-1) - y(n-2) + x(n-1) + x(n-2)$$

$$y(n) - 2y(n-1) + y(n-2) = x(n-1) + x(n-2)$$

$$Y(z)(1 - 2z^{-1} + z^{-2}) = X(z)(z^{-1} + z^{-2})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 - 2z^{-1} + z^{-2}} \quad (\times z^2)$$

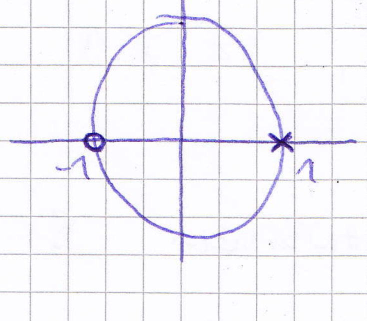
$$H(z) = \frac{z+1}{z^2 - 2z + 1}$$

* The roots of the numerator are the **zeros**,

$$z+1 = 0 \Rightarrow z = -1 \quad (\text{o})$$

* The roots of the denominator are the **poles**,

$$(z-1)^2 = 0 \Rightarrow z_{1,2} = 1 \quad (\times)$$



* If the ROC includes the poles, the system is stable

④ FIND THE ZEROS OF THE TRANSFER FUNCTION OF

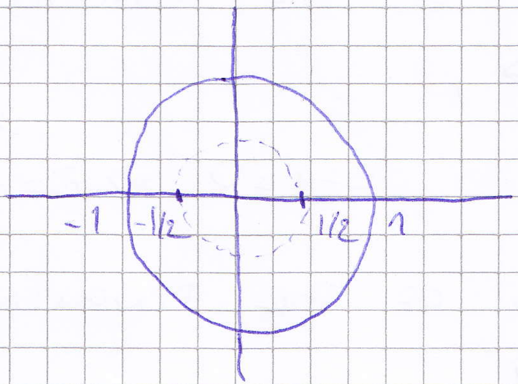
$$y(n) = x(n) - \frac{1}{4} x(n-2)$$

$$\Rightarrow \quad Y(z) \cdot z^{-0} = X(z)\left(z^{-0} - \frac{1}{4} z^{-2}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - \frac{z^{-2}}{4}$$

THEN THE ZEROS ARE,

$$1 - \frac{z^{-2}}{4} = 0 \Rightarrow z^{-2} = 4 \Rightarrow z_{1,2} = \pm \frac{1}{2}$$



⑤ DETERMINE THE SIGNAL $x(n)$ WHEN,

$$X(z) = \frac{z^2}{z^2 - z + 1}, \quad |z| > 1$$

* INVERSE Z-TRANSFORM ?

* USE TRANSFER PAIR

$$\boxed{\frac{z}{z-a} \longleftrightarrow a^n}$$

* PARTIAL FRACTIONS

$$\frac{z}{(z-p_1)(z-p_2)} = \frac{A}{z-p_1} + \frac{B}{z-p_2} \qquad (* z)$$

$$\underbrace{\frac{z^2}{(z-p_1)(z-p_2)}}_{?} = \frac{Az}{z-p_1} + \frac{Bz}{z-p_2}$$

- COMPLETING THE SQUARE

$$z^2 - z + 1 = 0$$

$$\left(z - \frac{1}{2}\right)^2 + 1 - \frac{1}{4} = 0$$

$$\left(z - \frac{1}{2}\right)^2 = -\frac{3}{4}$$

$$z - \frac{1}{2} = \pm\sqrt{-\frac{3}{4}}$$

$$z - \frac{1}{2} = \pm j\frac{\sqrt{3}}{2}$$

$$\Rightarrow z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$\left.\begin{aligned} \cos\omega &= \frac{1}{2} \\ \sin\omega &= \frac{\sqrt{3}}{2} \end{aligned}\right\} \omega = \frac{\pi}{3}$$

$$\Rightarrow \boxed{z = e^{\pm j\frac{\pi}{3}}}$$

THEN

$$X(z) = \frac{z^2}{(z-e^{j\frac{\pi}{3}})(z-e^{-j\frac{\pi}{3}})}$$

$$X(z) = \frac{Az}{z-e^{j\frac{\pi}{3}}} + \frac{Bz}{z-e^{-j\frac{\pi}{3}}}$$

$$\Rightarrow \quad z^2 = A.z(z-e^{-j\frac{\pi}{3}}) + B.z(z-e^{j\frac{\pi}{3}})$$

$$z^2 = Az^2 - Az.e^{-j\frac{\pi}{3}} + Bz^2 - Bz.e^{j\frac{\pi}{3}}$$

$$z^2 = z^2(A+B) + z(-A.e^{-j\frac{\pi}{3}} - B.e^{j\frac{\pi}{3}})$$

COEFFICIENTS FOR z^2 : $1 = A+B$ (1)

COEFFICIENTS FOR z^1 : $0 = -Ae^{-j\frac{\pi}{3}} - Be^{j\frac{\pi}{3}}$

$$Be^{j\frac{\pi}{3}} = -Ae^{-j\frac{\pi}{3}}$$

$$B = -\frac{Ae^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}}} \quad (2)$$

COEFFICIENTS FOR z^0 : $0 = 0$

THEN,

(2) → (1) $$1 = A - A\frac{e^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}}}$$

$$A\left(1 - \frac{e^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}}}\right) = 1$$

$$A = \frac{1}{1 - \frac{e^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}}}} = \frac{1}{\frac{e^{j\frac{\pi}{3}} - e^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}}}} = \boxed{\frac{e^{j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}} - e^{-j\frac{\pi}{3}}}}$$

① $B = 1 - A = \dfrac{\cancel{e^{j\frac{\pi}{3}}} - e^{-j\frac{\pi}{3}} - \cancel{e^{j\frac{\pi}{3}}}}{e^{j\frac{\pi}{3}} - e^{-j\frac{\pi}{3}}} = \boxed{\dfrac{e^{-j\frac{\pi}{3}}}{e^{j\frac{\pi}{3}} - e^{-j\frac{\pi}{3}}}}$

$$X(z) = \frac{Az}{z - e^{j\frac{\pi}{3}}} + \frac{Bz}{z - e^{-j\frac{\pi}{3}}}$$

$$\frac{z}{z-a} \longleftrightarrow a^n$$

$$\Rightarrow \boxed{x(n) = A \cdot e^{\frac{j\pi n}{3}} + B \cdot e^{-\frac{j\pi n}{3}}}$$

BUT,

$$e^{\pm j\frac{\pi}{3}} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

THEN,

$$A = \frac{\frac{1}{2} + j\frac{\sqrt{3}}{2}}{\cancel{\frac{1}{2}} + j\frac{\sqrt{3}}{2} - \cancel{\frac{1}{2}} + j\frac{\sqrt{3}}{2}} = \frac{1 + j\sqrt{3}}{j\sqrt{3}} = \frac{1}{j\sqrt{3}} + 1 = \boxed{1 - \frac{j}{\sqrt{3}}}$$

$$B = \frac{\frac{1}{2} - j\frac{\sqrt{3}}{2}}{\cancel{\frac{1}{2}} - j\frac{\sqrt{3}}{2} - \cancel{\frac{1}{2}} - j\frac{\sqrt{3}}{2}} = \frac{1 - j\sqrt{3}}{-j\sqrt{3}} = 1 - \frac{1}{j\sqrt{3}} = \boxed{1 + \frac{j}{\sqrt{3}}}$$

$$\Rightarrow x(n) = \left(1 - \frac{j}{\sqrt{3}}\right) \cdot e^{\frac{j\pi n}{3}} + \left(1 + \frac{j}{\sqrt{3}}\right) \cdot e^{-\frac{j\pi n}{3}}$$

$$x(n) = e^{\frac{j\pi n}{3}} - \frac{j e^{\frac{j\pi n}{3}}}{\sqrt{3}} + e^{-\frac{j\pi n}{3}} + \frac{j \cdot e^{-\frac{j\pi n}{3}}}{\sqrt{3}}$$

$$x(n) = \underbrace{\left(e^{\frac{j\pi n}{3}} + e^{-\frac{j\pi n}{3}}\right)}_{2\cos\left(\frac{\pi n}{3}\right)} - \frac{j}{\sqrt{3}} \underbrace{\left(e^{\frac{j\pi n}{3}} - e^{-\frac{j\pi n}{3}}\right)}_{2j \cdot \sin\left(\frac{\pi n}{3}\right)}$$

$$x(n) = 2\cos\left(\frac{\pi n}{3}\right) + \frac{2}{\sqrt{3}}\sin\left(\frac{\pi n}{3}\right)$$