SQNR for opt. Quantization (large R)

Δ_K is small $\rightarrow$ $\Delta_K = x_K - x_{K-1}$; $P_K = P_r(x \in I_K)$

$p(x=y_K) \triangleq \dfrac{P_K}{\Delta_K}$ [$\because$ large R and Δ_K small]

$$\sigma_q^2 = \sum_{K=1}^{L} \int_{x_{K-1}}^{x_K} (x-y_K)^2 \, p_X(x)\, dx$$

$$= \sum_{K=1}^{L} \int_{x_{K-1}}^{x_K} (x-y_K)^2 \frac{P_K}{\Delta_K}\, dx = \sum_{K=1}^{L} \frac{P_K}{\Delta_K} \int_{x_{K-1}}^{x_K} (x-y_K)^2\, dx .$$

$$\frac{\partial \sigma_q^2}{\partial y_K} = 0 \Rightarrow \frac{P_K}{\Delta_K} \int_{x_{K-1}}^{x_K} \{-2(x-y_K)\}\, dx = 0$$

or, $$\int_{x_{K-1}}^{x_K} x\, dx = y_K \int_{x_{K-1}}^{x_K} dx$$

or, $$\frac{1}{2}\left(x_K^2 - x_{K-1}^2\right) = y_K\left(x_K - x_{K-1}\right)$$

or, $$y_K = \frac{1}{2}\left(x_K + x_{K-1}\right) = x_{K-1} + \frac{\Delta_K}{2}$$

$$\sigma_q^2 = \sum_{K=1}^{L} \frac{P_K}{\Delta_K} \int_{x_{K-1}}^{x_{K-1}+\Delta_K} \left(x - \left(x_{K-1} + \frac{\Delta_K}{2}\right)\right)^2 dx$$

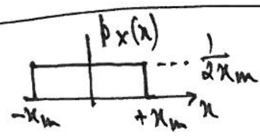
Let, $r = x - \left(x_{K-1} + \frac{\Delta_K}{2}\right)$

$$= \sum_{K=1}^{L} \frac{P_K}{\Delta_K} \int_{-\frac{\Delta_K}{2}}^{\frac{\Delta_K}{2}} r^2\, dx$$

$$= \sum_{K=1}^{L} \frac{P_K}{\Delta_K} \frac{1}{3} \left| r^3 \right|_{-\Delta_K/2}^{\Delta_K/2} = \sum_{K=1}^{L} \frac{P_K}{\Delta_K} \cdot \frac{1}{3}\left(\frac{\Delta_K^3}{8} + \frac{\Delta_K^3}{8}\right)$$

$$= \frac{1}{12} \sum_{K=1}^{L} P_K \Delta_K^2 .$$

Now, we take uniform distribution



$p_x(y_K) = \dfrac{1}{2x_m}$

$\Delta_K = \dfrac{2x_m}{L} = \dfrac{2x_m}{2^R}$.

$$\therefore \sigma_{q,opt}^2 = \frac{1}{12}\sum_{K=1}^{L} P_K \Delta_K^2 = \frac{1}{12}\sum_{K=1}^{L} p_x(y_K)\,\Delta_K^3 = \frac{1}{12}\sum_{K=1}^{L} \frac{1}{2x_m}\cdot\left(\frac{2x_m}{L}\right)^3 = \frac{1}{12}\cdot L \cdot \frac{1}{2x_m}\cdot\frac{2^3 x_m^3}{L^3}$$

$$= \frac{1}{3}\cdot\frac{x_m^2}{L^2}$$

6 dB law

$\sigma_x^2 = \dfrac{x_m^2}{3}$

$$\text{SQNR} = \frac{\sigma_x^2}{\sigma_q^2} = \frac{(x_m^2/3)}{\left(\frac{1}{3}\frac{x_m^2}{L^2}\right)} = L^2 = 2^{2R}$$

$$\text{SQNR}\big|_{dB} = 10\log_{10} L^2 = 10\log_{10} 2^{2R} = 6{\cdot}02R \text{ dB} \quad : 6 \text{ dB law}$$

For arbitrary pdf, high rate theory says

$$SQNR|_{dB} = 6{\cdot}02R - 10\log_{10}\epsilon_*^2$$, where ϵ_*^2 is a pdf dependent factor.

Uniform Quantization

Input signal: $x_{min} \leq x \leq x_{max}$ with $x_{min} = -x_{max}$.

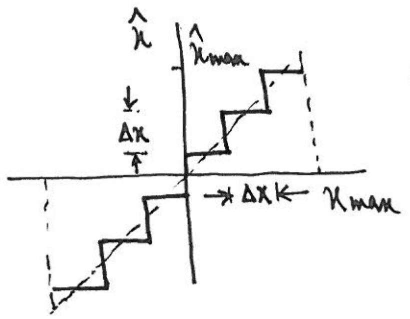
Quantized signal: (output) $\hat{x}_{min} \leq \hat{x} \leq \hat{x}_{max}$ with $\hat{x}_{min} = -\hat{x}_{max}$.

$K = 2^{w} \rightarrow$ ~~steps or~~ quantization levels

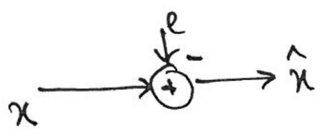
uniform intervals $\Delta x = \dfrac{2x_{max}}{2^{w}} = \dfrac{x_{max}}{2^{w-1}}$.

Δx is called quantizer step size.

$$\hat{x} = f(x) \in \left\{ \hat{x}_i = \pm (2i-1)\cdot\frac{\Delta x}{2} \right\}$$

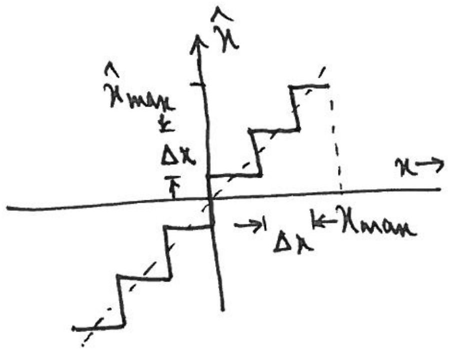


Q. Noise: $e = x - \hat{x}$



Quantization noise is also assumed to be a random variable.

- **Uniform midrise Quantizer**

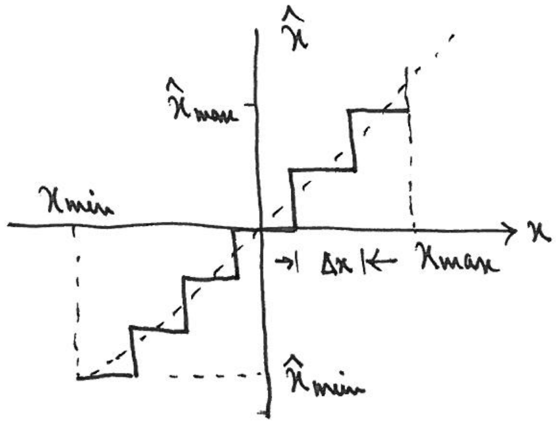


- symmetric
- $\hat{x}_{max} = (2^{w}-1)\,\frac{\Delta x}{2}$
- $\hat{x} \in \{\pm(2i-1)\,.\,\frac{\Delta x}{2}\}$
- $x_{max} = 2^{w}\,.\,\frac{\Delta x}{2}$
- $x_{min} = -\,x_{max}$
- $\mathcal{E}\{e^2\} = \frac{\Delta x^2}{12}$

disadvantage : no null quantized value.

$$\# \quad \hat{x} = \mathrm{sign}(x)\,.\,\Delta x\,.\left[\mathrm{int}\left(\frac{|x|}{\Delta x}\right) + 0{\cdot}5\right]$$

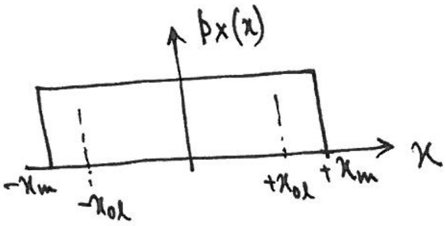
- **Uniform midtread Quantizer**



- $\mathcal{E}\{e^2\} = \frac{\Delta x^2}{12}$.
- $\hat{x}_{max} = 2^{w}\,.\,\frac{\Delta x}{2} - \Delta x$
- $\hat{x} \in \{\pm\, i\,.\,\Delta x\}$
- $x_{max} = (2^{w}-1)\,.\,\frac{\Delta x}{2}$
- $x_{min} = -(2^{w}+1)\,.\,\frac{\Delta x}{2}$
- $\mathcal{E}\{e^2\} = \frac{\Delta x^2}{12}$.

$$\# \quad \hat{x} = \mathrm{sign}(x)\,.\,\Delta x\,.\,\mathrm{int}\left(\frac{|x|}{\Delta x} + 0{\cdot}5\right)$$

Non-Uniform Quantization (Overload error)



$$p_x(x) = \text{uniform} = \frac{1}{2x_m}$$

$$\sigma_q^2 = \sigma_{q,\,gran}^2 + \sigma_{q,\,ovl}^2 .$$

$$\sigma_{q,\,gran}^2 = \frac{\Delta^2}{12} \quad \text{where} \quad \Delta = \frac{2x_{ol}}{L}, \quad L = 2^R$$

$$\sigma_{q,\,ovl}^2 = 2\int_{x_{ol}}^{x_m} (x - x_{ol})^2 \, p_x(x)\, dx$$

$$= 2 \cdot \frac{1}{2x_m} \int_{x_{ol}}^{x_m} (x - x_{ol})^2 \, dx$$

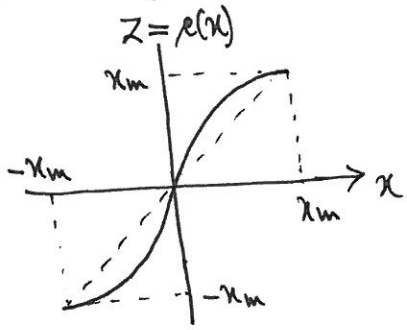
$$= \frac{1}{x_m} \cdot \frac{1}{3} \left| (x - x_{ol})^3 \right|_{x_{ol}}^{x_m}$$

$$= \frac{1}{3x_m} (x_m - x_{ol})^3$$

$$\sigma_q^2 = \sigma_{q,\,gran}^2 + \sigma_{q,\,ovl}^2 = \frac{4x_{ol}^2/L^2}{12} + \frac{1}{3x_m}(x_m - x_{ol})^3$$

Note: This is very important to choose x_{ol}.

Companding



- x is transformed to a Z random variable. $Z = c(x)$.
- uniform quantization on Z.
  $\rightarrow$ leads to non-uniform quantization in x.
- $\hat{x} = c^{-1}\left[Q\left[c(x)\right]\right]$

Large $R \Rightarrow$ large $L \Rightarrow$ small Δ_K, but Δ_K is non-uniform.

$$c'(x) = \frac{d}{dx} c(x) = \frac{\frac{2x_m}{L} \quad (\text{over } z)}{\Delta_K \quad (\text{over } x)}$$

Optimum Non-U Quant

$$\sigma_q^2 = \frac{1}{12} \sum_{K=1}^{L} P_K \Delta_K^2 = \frac{1}{12} \sum_{K=1}^{L} P_K \cdot \left(\frac{2x_m/L}{c'(x)} \right)^2$$

$$= \frac{x_m^2}{3L^2} \sum_{K=1}^{L} \int_{x_{k-1}}^{x_K} p_X(x) \left[c'(x)\right]^{-2} dx \qquad [\because \text{large } R]$$

$$= \frac{x_m^2}{3L^2} \int_{-x_m}^{x_m} p_X(x) \left[c'(x)\right]^{-2} dx$$

Now, $$SQNR = \frac{\sigma_x^2}{\sigma_q^2} = \frac{\int_{-x_m}^{x_m} x^2 p_X(x)\, dx}{\frac{x_m^2}{3L^2} \int_{-x_m}^{x_m} p_X(x) \left[c'(x)\right]^{-2} dx}$$

- If we choose $c'(x) = \frac{a}{x}$ then SQNR is independent of signal variance.

$$SQNR = \frac{3L^2 a^2}{x_m^2} \quad \text{and} \quad c(x) = a \ln x + k', \; x > 0$$

and ~~satisfying the boundary condition~~.

Boundary conditions $c(0) = 0$
$c(x_m) = x_m$

- The function $\ln(x)$ is only defined for positive values and diverges for $x \to 0$. So, use of pure logarithmic compression is not possible.

- So, we need some approximations (pragmatic modification).

- For the fixed (wire-line) digital telephone networks in Europe, the compander is A-law compander. (normalized)

$$g_A(x) = \begin{cases} \text{sign}(x) \cdot \dfrac{1 + \ln(A|x|)}{1 + \ln A} & \dfrac{1}{A} < |x| \leqslant \pm 1 \\ \dfrac{Ax}{1 + \ln(A)} & -\dfrac{1}{A} \leqslant x \leqslant +\dfrac{1}{A} \end{cases}$$