EL2620 Nonlinear Control

Lecture 14

KTH Electrical Engineering

- Summary and repetition
- Courses in control

Exam

- Sat Jan 18 2014, 14.00-19.00
- Regular written exam with five problems
- You may bring lecture notes, Glad & Ljung “Reglerteknik” (”basic control book”), and TEFYMA or BETA
  - no other material: textbooks, exercises, calculators etc.
  - any other basic control book must be approved by me *before* the exam.
- See course homepage for old exams
- Q&A session before the exam: Mon Jan 13 10-12 (see homepage)

Question 1

What's on the exam?

- Nonlinear models: equilibria, phase portaits, linearization and classification
- Lyapunov stability (local and global), LaSalle
- Small Gain Theorem, Circle Criterion, Passivity Theorem
- Describing functions
- Compensating static nonlinearities
- Exact feedback linearization, input-output linearization, zero dynamics
- Sliding modes, equivalent controls
- Lyapunov based design: back-stepping
- Nonlinear controllability
- Optimal control

Question 2

What design method should I use in practice?

The answer is highly problem dependent. Possible (learning) approach:

- Start with the simplest:
  - linear methods (loop shaping, state feedback, ...)
- Evaluate:
  - strong nonlinearities (under feedback!)?
  - varying operating conditions?
  - analyze and simulate with nonlinear model
- Some nonlinearities to compensate for?
  - saturations, valves etc
- Is the system generically nonlinear? E.g, $\dot{x} = xu$

Question: Can you repeat nonlinear controllability?

The system

$$\dot{x} = f(x, u)$$

is **controllable** if for any x^0, x^1 there exists $T > 0$ and $u : [0, T] \to \mathbb{R}$ such that $x(0) = x^0$ and $x(T) = x^1$.

- Locally controllable if linearization is controllable.
- May be controllable even if linearization not controllable

Lie Brackets

Lie bracket between vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field defined by

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

- **Controllability theorem**

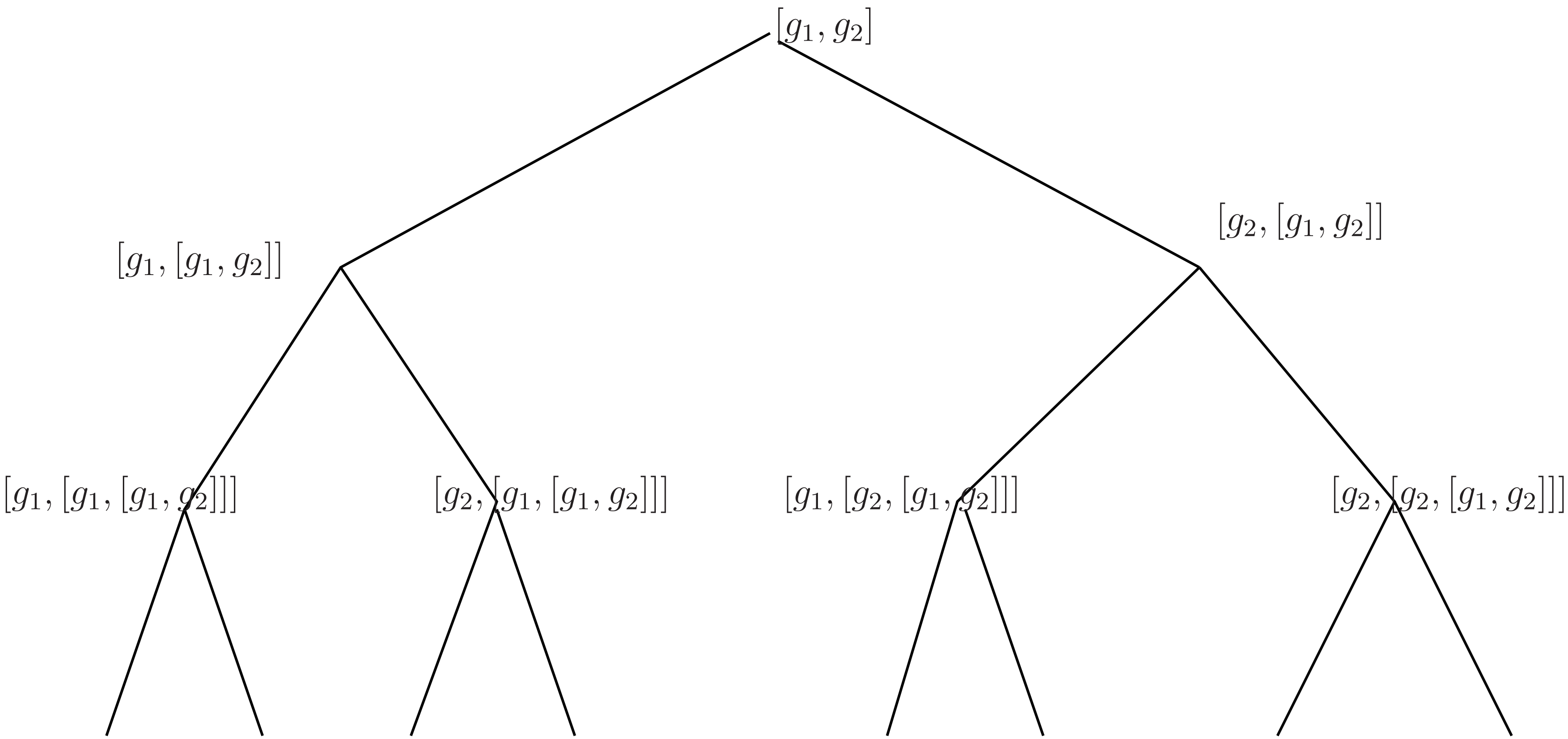
$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

is controllable if the Lie bracket tree (together with g_1 and g_2) spans $\mathbb{R}^n$ for all x

Remark:

– The system can be steered in any direction of the Lie bracket tree

The Lie Bracket Tree



Question 3

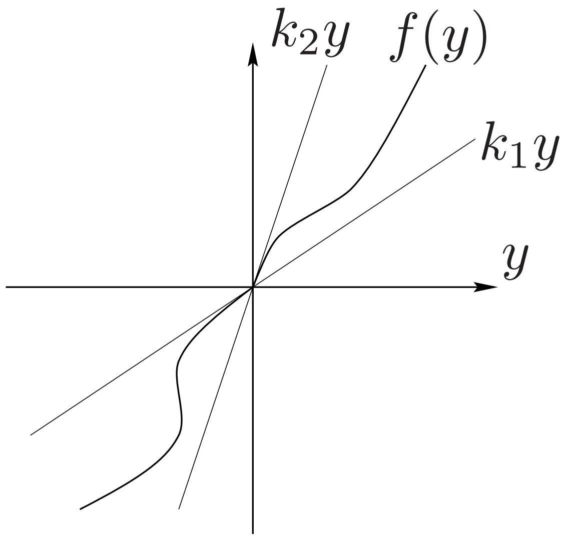
Can a system be proven stable with the Small Gain Theorem and unstable with the Circle Criterion?

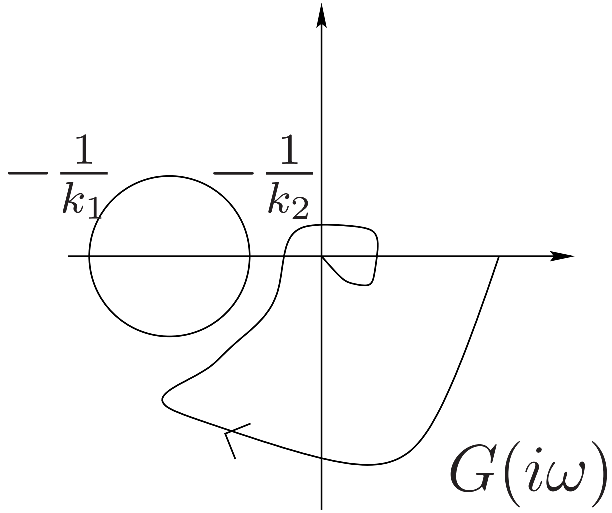
- No, the Small Gain Theorem, Passivity Theorem and Circle Criterion all provide only **sufficient conditions for stability**.
- But, if one method does not prove stability, another one may.
- Since they do not provide necessary conditions for stability, none of them can be used to prove instability.

Question 4

Can you review the Circle Criterion? What about $k_1 < 0 < k_2$?

The Circle Criterion





Theorem Consider a feedback loop with $y = Gu$ and $u = -f(y)$. Assume $G(s)$ is stable and that

$$k_1 \leq f(y)/y \leq k_2.$$

If the Nyquist curve of $G(s)$ stays on the correct side of the circle defined by the points $-1/k_1$ and $-1/k_2$, then the closed-loop system is BIBO stable.

The different cases

Stable system G

1. $0 < k_1 < k_2$: Stay outside circle
2. $0 = k_1 < k_2$: Stay to the right of the line Re $s = -1/k_2$
3. $k_1 < 0 < k_2$: Stay inside the circle

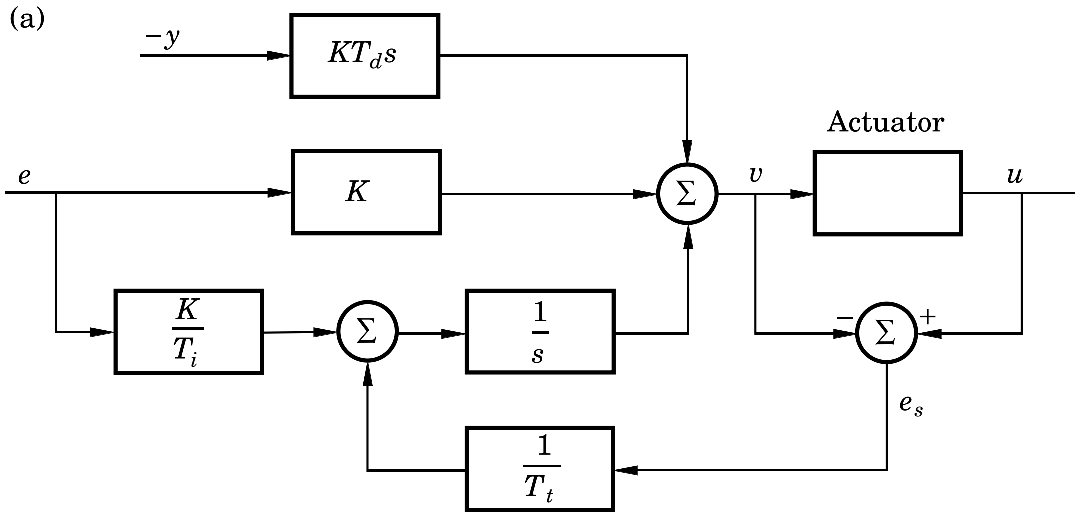
Other cases: Multiply f and G with -1.

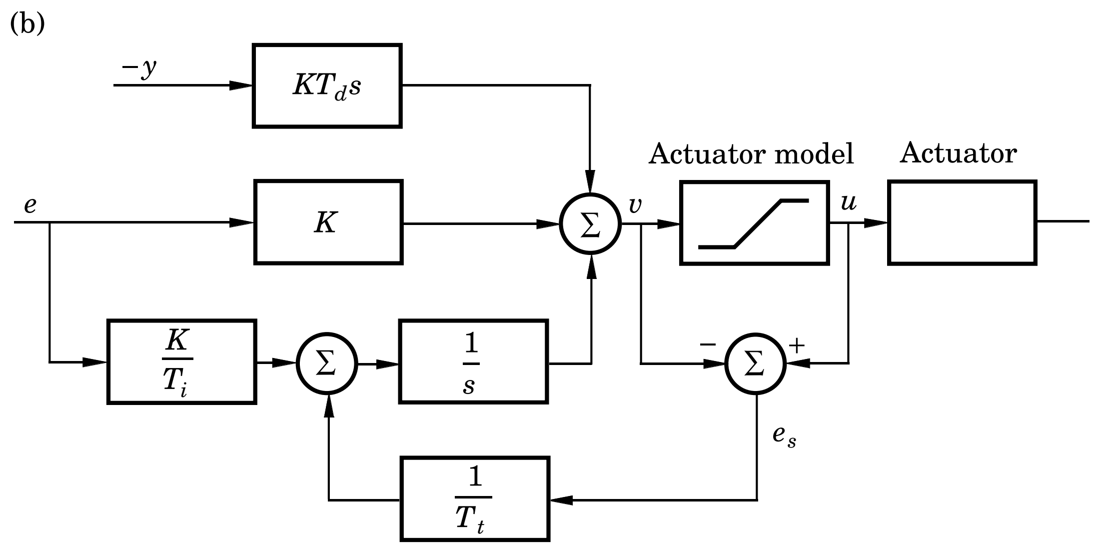
Only Case 1 and 2 studied in lectures. Only G stable studied.

Question 6

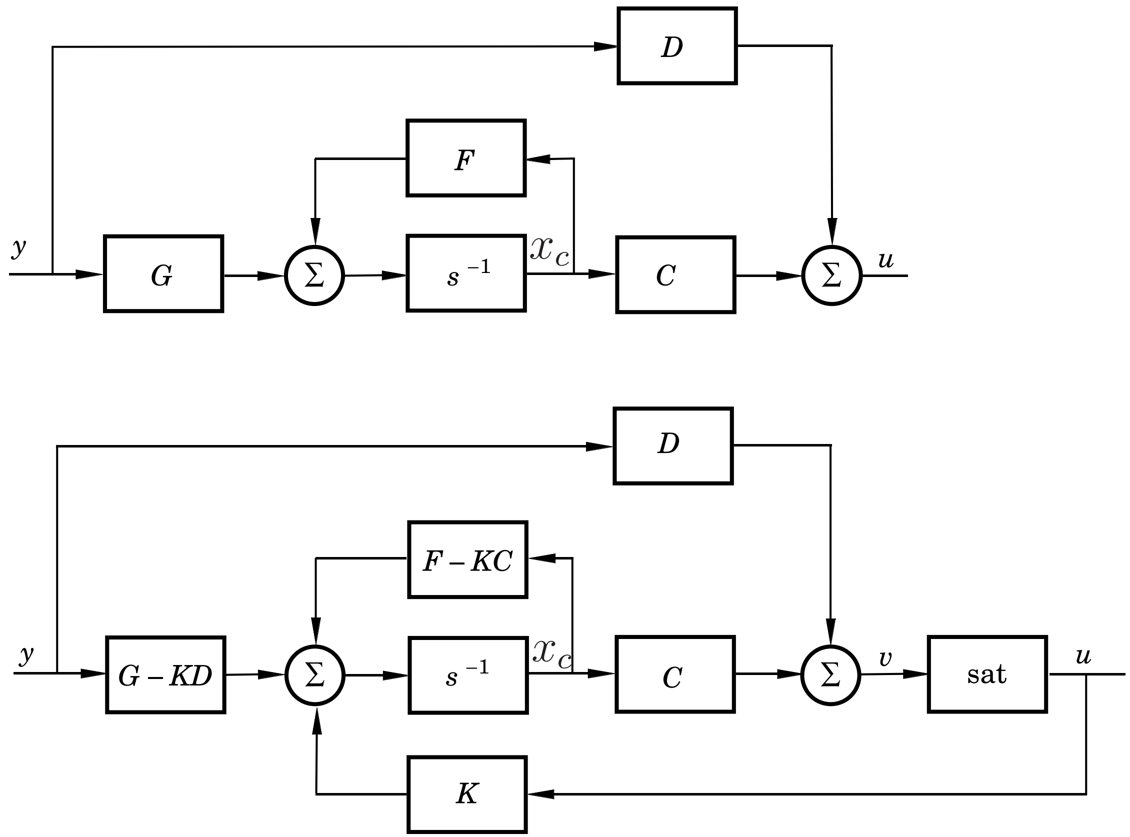
Please repeat antiwindup

Tracking PID





Antiwindup—General State-Space Model



Choose K such that $F - KC$ has stable eigenvalues.

Question 5

Please repeat Lyapunov theory

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

locally stable, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

locally asymptotically stable, if locally stable and

$$\|x(0)\| < r \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0$$

globally asymptotically stable, if asymptotically stable for all $x(0) \in \mathbf{R}^n$.

Lyapunov Theorem for Local Stability

Theorem Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$. Assume that $V : \Omega \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
- $\dot{V}(x) \leq 0$ along all trajectories in Ω

then $x = 0$ is locally stable. Furthermore, if

- $\dot{V}(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

Lyapunov Theorem for Global Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$. Assume that $V : \mathbf{R}^n \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \neq 0$
- $\dot{V}(x) < 0$ for all $x \neq 0$
- $V(x) \to \infty$ as $\|x\| \to \infty$

then $x = 0$ is globally asymptotically stable.

LaSalle's Theorem for Global Stability

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $\mathbb{C}^1$ function $V : R^n \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \neq 0$

(3) $\dot{V}(x) \leq 0$ for all x

(4) $V(x) \to \infty$ as $\|x\| \to \infty$

(5) The only solution of $\dot{x} = f(x)$ such that $\dot{V}(x) = 0$ is $x(t) = 0$ for all t

then $x = 0$ is globally asymptotically stable.

LaSalle's Invariant Set Theorem

Theorem Let $\Omega \in \mathbf{R}^n$ be a bounded and closed set that is invariant with respect to

$$\dot{x} = f(x).$$

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a C^1 function such that $\dot{V}(x) \leq 0$ for $x \in \Omega$. Let E be the set of points in Ω where $\dot{V}(x) = 0$. If M is the largest invariant set in E, then every solution with $x(0) \in \Omega$ approaches M as $t \to \infty$

Remark: a **compact set** (bounded and closed) is obtained if we e.g., consider

$$\Omega = \{x \in \mathbf{R}^n | V(x) \leq c\}$$

and V is a positive definite function

Relation to Poincare-Bendixson Theorem

Poincare-Bendixson *Any orbit of a continuous 2nd order system that stays in a compact region of the phase plane approaches its* ω*-limit set, which is either a fixed point, a periodic orbit, or several fixed points connected through homoclinic or heteroclinic orbits*

In particular, if the compact region does not contain any fixed point then the ω-limit set is a limit cycle

Example: Pendulum with friction

$$\dot{x}_1 = x_2 \,, \quad \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

$$V(x) = \frac{g}{l}(1-\cos x_1) + \frac{1}{2}x_2^2 \quad \Rightarrow \quad \dot{V} = -\frac{k}{m}x_2^2$$

- We can not prove global asymptotic stability; why?
- The set $E = \{(x_1, x_2) | \dot{V} = 0\}$ is $E = \{(x_1, x_2) | x_2 = 0\}$
- The invariant points in E are given by $\dot{x}_1 = x_2 = 0$ and $\dot{x}_2 = 0$. Thus, the largest invariant set in E is

$$M = \{(x_1, x_2) | x_1 = k\pi, x_2 = 0\}$$

- The domain is compact if we consider $\Omega = \{(x_1, x_2) \in \mathbf{R}^2 | V(x) \le c\}$

- If we e.g., consider $\Omega : x_1^2 + x_2^2 \leq 1$ then $M = \{(x_1, x_2) | x_1 = 0, x_2 = 0\}$ and we have proven asymptotic stability of the origin.

Question 6

Please repeat the most important facts about sliding modes.

There are 3 essential parts you need to understand:

1. The sliding manifold
2. The sliding control
3. The equivalent control

Step 1. The Sliding Manifold S

Aim: we want to stabilize the equilibrium of the dynamic system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^1$$

Idea: use u to force the system onto a *sliding manifold* S of dimension $n-1$ in finite time

$$S = \{x \in \mathbb{R}^n | \sigma(x) = 0\} \quad \sigma \in R^1$$

and make S invariant

If $x \in \mathbb{R}^2$ then S is $\mathbb{R}^1$, i.e., a curve in the state-plane (phase plane).

Example

$$\dot{x}_1 = x_2(t)$$
$$\dot{x}_2 = x_1(t)x_2(t) + u(t)$$

Choose S for desired behavior, e.g.,

$$\sigma(x) = ax_1 + x_2 = 0 \quad \Rightarrow \quad \dot{x}_1 = -ax_1(t)$$

Choose large a: fast convergence along sliding manifold

Step 2. The Sliding Controller

Use Lyapunov ideas to design $u(x)$ such that S is an attracting invariant set

Control Lyapunov Function $V(x) = 0.5\sigma^2$ yields $\dot{V} = \sigma\dot{\sigma}$

For 2nd order system $\dot{x}_1 = x_2$, $\dot{x}_2 = f(x) + g(x)u$ and $\sigma = x_1 + x_2$ we get

$$\dot{V} = \sigma\left(x_2 + f(x) + g(x)u\right) < 0 \quad \Leftarrow \quad u = -\frac{f(x) + x_2 + sgn(\sigma)}{g(x)}$$

<u>Example:</u> $f(x) = x_1x_2,\ g(x) = 1,\ \sigma = x_1 + x_2$, yields

$$u = -x_1x_2 - x_2 - sgn(x_1 + x_2)$$

Step 3. The Equivalent Control

When trajectory reaches sliding mode, i.e., $x \in S$, then u will chatter (high frequency switching).

However, an equivalent control $u_{eq}(t)$ that keeps $x(t)$ on S can be computed from $\dot{\sigma} = 0$ when $\sigma = 0$

<u>Example:</u>

$$\dot{\sigma} = \dot{x}_1 + \dot{x}_2 = x_2 + x_1 x_2 + u_{eq} = 0 \quad \Rightarrow \quad u_{eq} = -x_2 - x_1 x_2$$

Thus, the sliding controller will take the system to the sliding manifold S in finite time, and the equivalent control will keep it on S.

Note!

Previous years it has often been assumed that the sliding mode control always is on the form

$$u = -sgn(\sigma)$$

This is OK, but is not completely general (see example)

Question 7

Can you repeat backstepping?

Backstepping Design

We are concerned with finding a stabilizing control $u(x)$ for the system

$$\dot{x} = f(x, u)$$

General Lyapunov control design: determine a Control Lyapunov function $V(x, u)$ and determine $u(x)$ so that

$$V(x) > 0 \,, \quad \dot{V}(x) < 0 \, \forall x \in \mathbb{R}^n$$

In this course we only consider $f(x, u)$ with a special structure, namely **strict feedback structure**

Strict Feedback Systems

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \dots, x_n) + g_n(x_1, \dots, x_n)u
\end{aligned}
$$

where $g_k \neq 0$

Note: $x_1, \dots, x_k$ do not depend on $x_{k+2}, \dots, x_n$.

The Backstepping Idea

Given a Control Lyapunov Function $V_1(x_1)$, with corresponding control $u = \phi_1(x_1)$, for the system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u$$

find a Control Lyapunov function $V_2(x_1, x_2)$, with corresponding control $u = \phi_2(x_1, x_2)$, for the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + u\end{aligned}$$

The Backstepping Result

Let $V_1(x_1)$ be a Control Lyapunov Function for the system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u$$

with corresponding controller $u = \phi(x_1)$.

Then $V_2(x_1, x_2) = V_1(x_1) + (x_2 - \phi(x_1))^2/2$ is a Control Lyapunov Function for the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + u\end{aligned}$$

with corresponding controller

$$u(x) = \frac{d\phi}{dx_1}\Big(f(x_1)+g(x_1)x_2\Big) - \frac{dV}{dx_1}g(x_1) - (x_k - \phi(x_1)) - f_2(x_1, x_2)$$

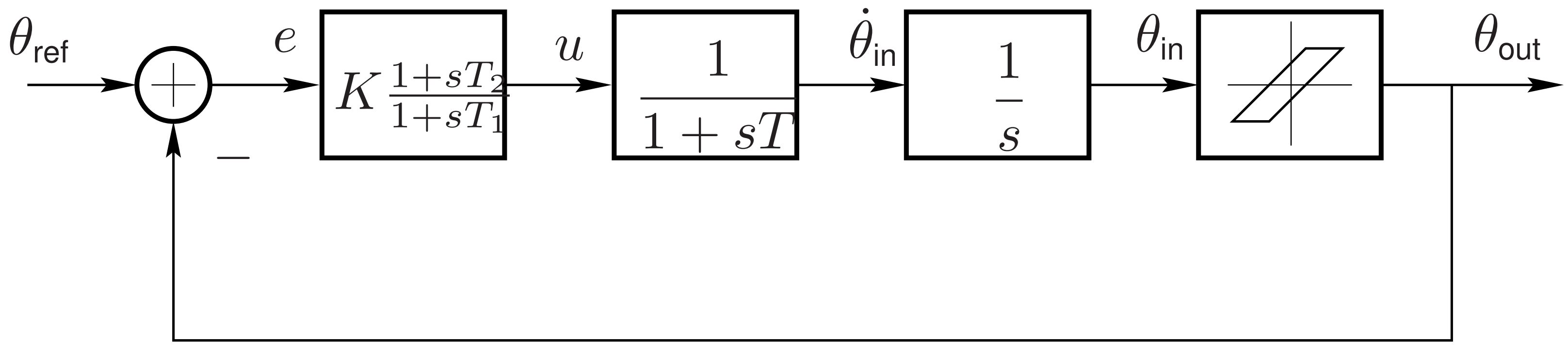
Question 8

Repeat backlash compensation

Backlash Compensation

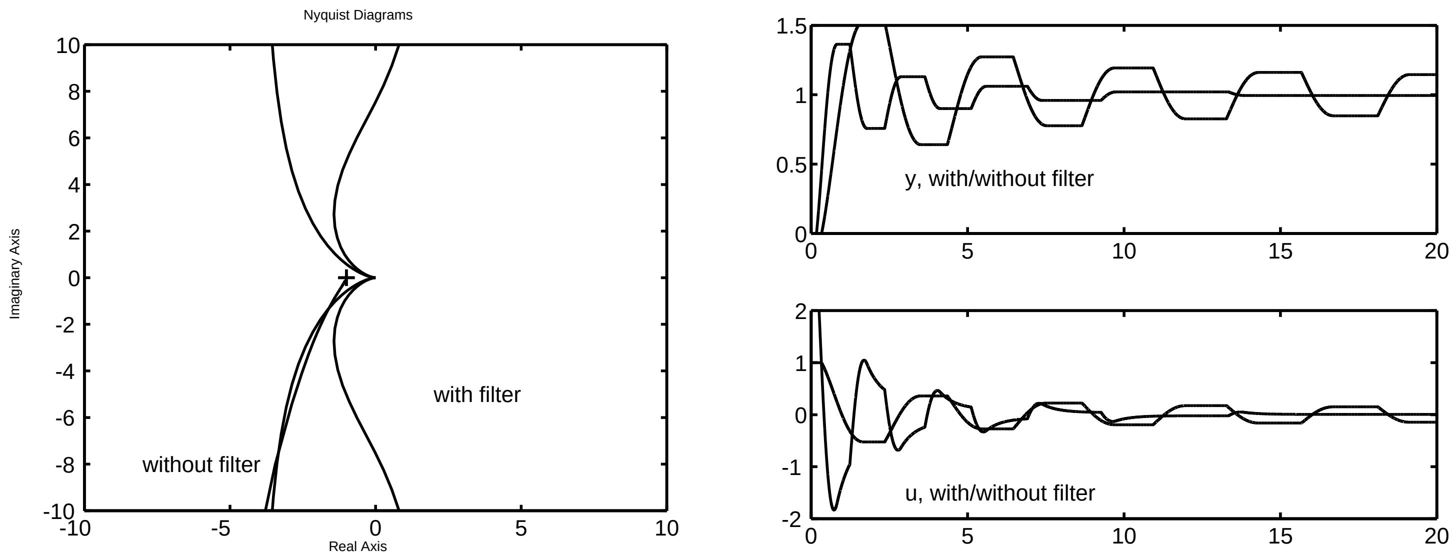
- Deadzone
- Linear controller design
- Backlash inverse

Linear controller design: Phase lead compensation



- Choose compensation $F(s)$ such that the intersection with the describing function is removed

$F(s) = K\frac{1+sT_2}{1+sT_1}$ with $T_1 = 0.5, T_2 = 2.0$:



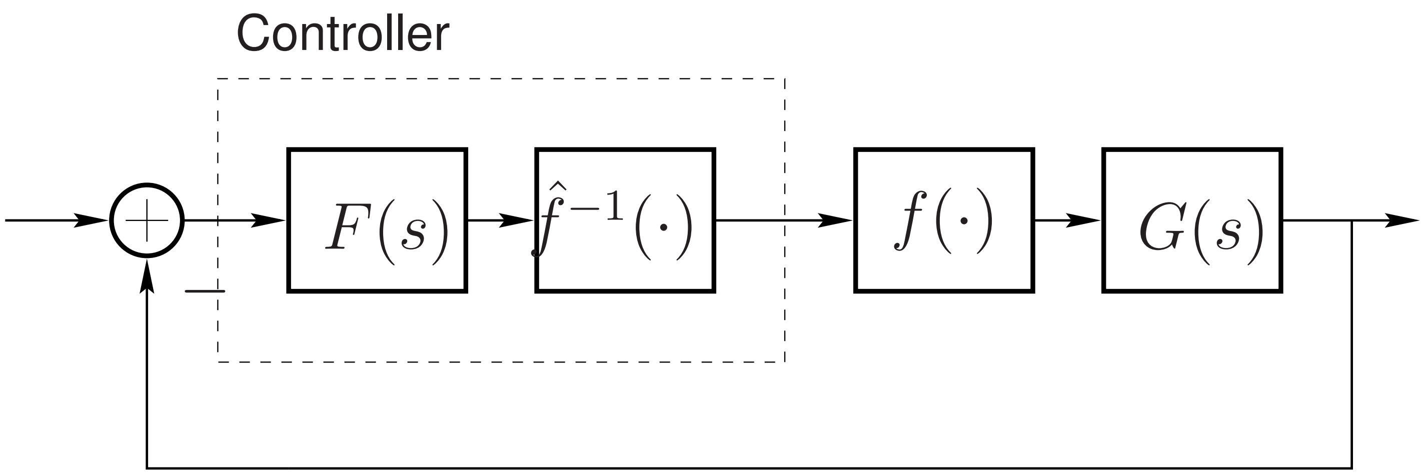
Oscillation removed!

Question 9

Can you repeat linearization through high gain feedback?

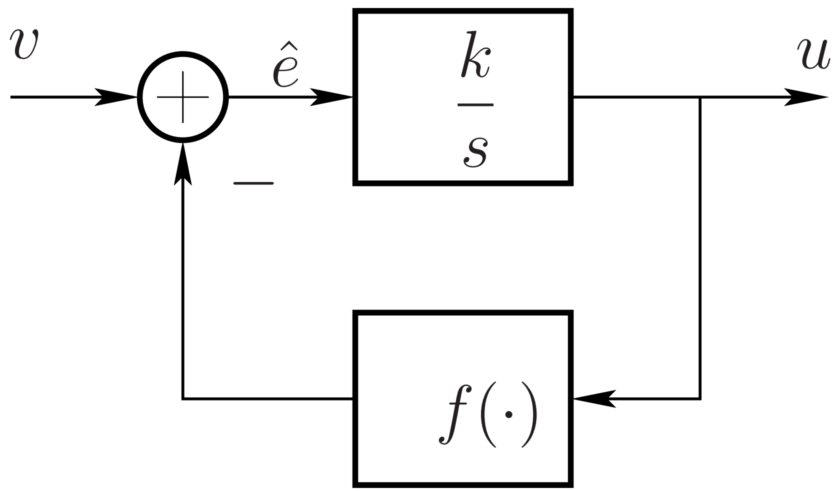
Inverting Nonlinearities

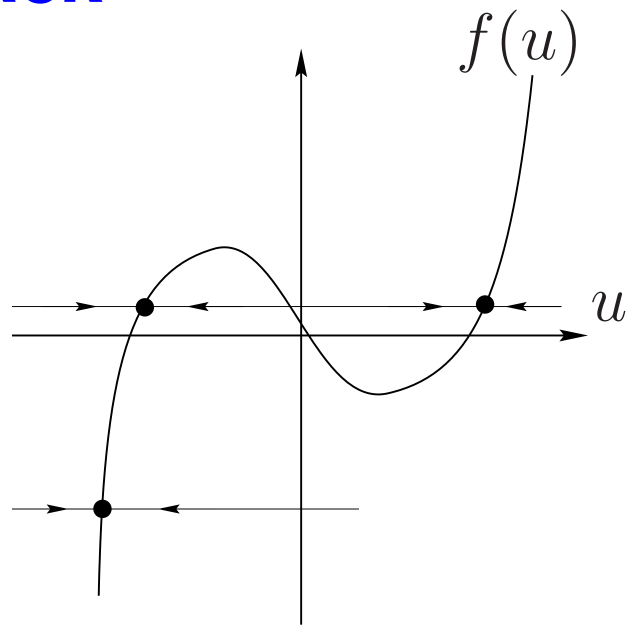
Compensation of static nonlinearity through inversion:



Should be combined with feedback as in the figure!

Remark: How to Obtain f^{-1} from f using Feedback





$$\hat{e} = \big(v - f(u)\big)$$

If $k > 0$ large and $df/du > 0$, then $\hat{e} \to 0$ and

$$0 = \big(v - f(u)\big) \quad \Leftrightarrow \quad f(u) = v \quad \Leftrightarrow \quad u = f^{-1}(v)$$

Question 10

What should we know about input–output stability?

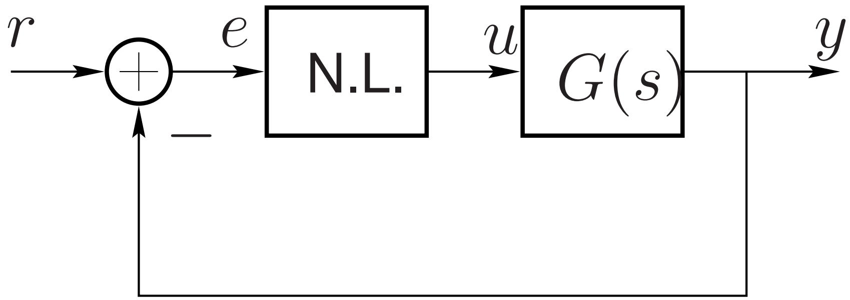
You should understand and be able to derive/apply

- System gain $\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2}$
- BIBO stability
- Small Gain Theorem
- Circle Criterion
- Passivity Theorem

Question 11

What about describing functions?

Idea Behind Describing Function Method



$e(t) = A \sin \omega t$ gives

$$u(t) = \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \sin[n\omega t + \arctan(a_n/b_n)]$$

If $|G(in\omega)| \ll |G(i\omega)|$ for $n \geq 2$, then $n = 1$ suffices, so that

$$y(t) \approx |G(i\omega)| \sqrt{a_1^2 + b_1^2} \sin[\omega t + \arctan(a_1/b_1) + \arg G(i\omega)]$$

Definition of Describing Function

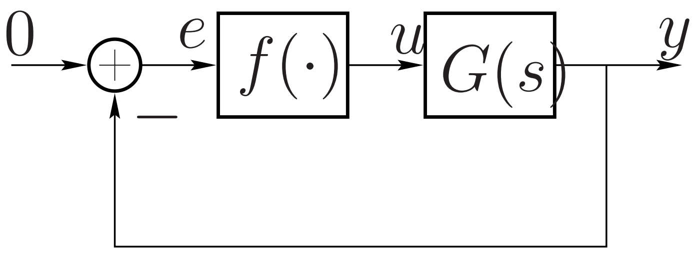
The **describing function** is

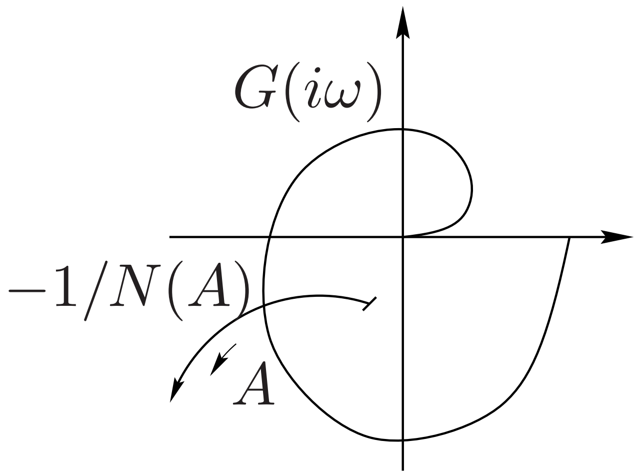
$$N(A,\omega) = \frac{b_1(\omega) + i a_1(\omega)}{A}$$

If G is low pass and $a_0 = 0$ then

$$\widehat{u}_1(t) = |N(A,\omega)| A \sin[\omega t + \arg N(A,\omega)] \approx u(t)$$

Existence of Periodic Solutions





$$y = G(i\omega)u = -G(i\omega)N(A)y \quad \Rightarrow \quad G(i\omega) = -\frac{1}{N(A)}$$

The intersections of the curves $G(i\omega)$ and $-1/N(A)$ give ω and A for a possible periodic solution.

QUESTIONS?

Some Other Courses in Control

- EL2450 Hybrid and Embedded Control Systems, per 3
- EL2520 Control Theory and Practice, Advanced Course, per 4
- EL1820 Modelling of Dynamical Systems, per 1
- EL2745 Principles of Wireless Sensor Networks, per 1
- EL2421 Project Course in Automatic Control, per 2

EL2450 Hybrid and Embedded Control Systems

Aim: course on analysis, design and implementation of control algorithms in networked and embedded systems.

- Period 3, 7.5 cr
- How are control systems realized:
  - computer-implementation of control algorithms
  - scheduling of real-time software
  - control over communication networks
- Lectures, exercises, homework, computer exercises

Contact: Dimos Dimarogonas `dimos@ee.kth.se`

EL2745 Principles of Wireless Sensor Networks

Aim: provide the participants with a basic knowledge of wireless sensor networks (WSN)

- Period 1, 7.5 cr
- THE INTERNET OF THINGS
  - essential tools within communication, control, optimization and signal processing needed to cope with WSN
  - design of practical WSNs
  - research topics in WSNs

Contact: Carlo Fischione `carlofi@kth.se`

EL2520 Control Theory and Practice, Advanced Course

Aim: provide an introduction to principles and methods in advanced control, especially multivariable feedback systems.

- Period 4, 7.5 cr
- Multivariable control:
  - linear multivariable systems
  - robustness and performance
  - synthesis of multivariable controllers: LQG, $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimization
  - real time optimization: Model Predictive Control (MPC)
- Lectures, exercises, labs, computer exercises

Contact: Mikael Johansson `mikaelj@kth.se`

EL2820 Modelling of Dynamic Systems

Aim: teach how to systematically build mathematical models of technical systems from physical laws and from measured signals.

- Period 1, 6 cr
- Model dynamical systems from
  - physics: lagrangian mechanics, electrical circuits etc
  - experiments: parametric identification, frequency response
- Computer tools for modeling, identification, and simulation
- Lectures, exercises, labs, computer exercises

Contact: Cristian Rojas, `crro@kth.se`

EL2421 Project Course in Control

Aim: provide practical knowledge about modeling, analysis, design, and implementation of control systems. Give some experience in project management and presentation.

- Period 2, 15 cr
- “From start to goal...”: apply the theory from other courses
- Team work
- Preparation for Master thesis project
- Project management (lecturers from industry)
- No regular lectures or labs

Contact: Jonas Mårtensson, `jonas1@kth.se`

Doing Master Thesis Project at KTH Control Lab

- Theory and practice
- Cross-disciplinary
- The research edge
- Collaboration with leading industry and universities
- Get insight in research and development

Hints:

- The topic and the results of your thesis are up to you
- Discuss with professors, lecturers, PhD and MS students
- Check old projects

Doing PhD Thesis Project at KTH Control Lab

- Intellectual stimuli
- Get paid for studying
- International collaborations and travel
- Competitive
- World-wide job market
- Research (60%), courses (30%), teaching (10%), fun (100%)
- 4-5 yr's to PhD (lic after 2-3 yr's)