Introduction

This homework mainly concerns modeling and analysis of hybrid control systems.

Two Tank System [20p]

In the first part of this homework, we will study the two tank system shown in Figure 1. W represents a pump that continuously pumps water into two tanks with a volume w. We try to keep the water level of both tanks $x_1, x_2$ above $r_1, r_2$, respectively. $v_1$ and $v_2$ stand for the outgoing volume of each tank.

![Figure 1: A two tank system with one pump alternating between the tanks.](image)

Questions

1. Try to model the two-tank system as a hybrid automaton with two discrete and two continuous states as illustrated in Figure 2. (refer to Lecture 10). [3p]

2. Assume the cross-sectional areas of the tanks are set to one and all parameters have standard units. Given the parameter settings as in
Table 1, model the system in MatLab/Simulink. Please enclose important parts of your code or simulink models to motivate your modeling method. [4p]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$x^0$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$w$</th>
<th>$v_1$</th>
<th>$v_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>(5 4)$^T$</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: Parameter values for the two tank system

3. Simulate the system you modeled and explain what happens during the simulation. When will there be transitions? When are the guard conditions true? Please use plots of the state trajectory and arguments from the theory of hybrid automata in your explanation. [5p]

From the reading material ([J] and [JLS]), we know that the tanks exhibit Zeno behavior if $\max(v_1,v_2) < w < v_1 + v_2$. The Zeno time in this case is given by

$$
\tau_{\infty} = \frac{x_1^0 + x_2^0 - r_1 - r_2}{v_1 + v_2 - w}.
$$

(1)

4. Change the inflow volume to $w = 4$ and simulate your model again. Plot the results and explain briefly what happens. Is Zeno behavior visible? Is the Zeno time as expected in Equation (1)? [5p]

5. Compare your simulation results with what you may expect in real-life two tank systems, and explain the possible reasons. [3p]

Newton’s Cradle [30 p]

In the second part, we will take into account the famous model of Newton’s cradle (see http://en.wikipedia.org/wiki/Newton’s_Cradle). As shown in Figure 3, in this case with three balls, each ball hangs in a rope. The ropes are parallel to each other when the three balls are still. Moreover, we consider totally elastic impacts in the sense that all energy and momentum is transferred from one to another when two balls collide. More specifically:
At $t = 0$, ball 1 is released from $\theta_1 = +\frac{\pi}{8}$ and it falls obeying the following rule

\[
\dot{\theta} = \omega \\
\dot{\omega} = -\frac{g}{l}\sin{\theta}
\]

When it reaches the position $\theta_1 = 0$ with a negative angular velocity $\omega_1$, it collides with ball 2 and exchanges angular velocity, i.e., $\omega_2 = \omega_1$ and $\omega_1 = 0$.

After that, ball 2 obtains the energy or momentum from ball 1, and it collides with ball 3 immediately and exchanges angular velocity, i.e., $\omega_3 = \omega_2$ and $\omega_2 = 0$. Thus ball 3 has the same angular velocity as ball 1 has when ball 1 collides with ball 2.

Then ball 3 swings up and follows the same dynamics as in Equation (2). Afterwards, the phenomenon described above is iterated.

Questions

6. Formally define the hybrid automaton modeling the Newton’s cradle. Use the notation in Lecture 10 or the Appendix. [6p]

Note: Ways to define the hybrid automaton may not be unique.

7. Set the rope length to be 10(m) and ball diameter to be 1(m). Model the process in MatLab/Simulink. Please enclose the important parts of your code or simulink models to motivate your modeling method. [6p]

Hint: You may need to add some margin when validating the guards, because a number equal to zero by calculation may not be exactly zero in the digital computation.

8. Now simulate the system to see how it works. Using the plots of the state trajectory and the hybrid model, explain what happens during the simulation. When will transitions take place? When are the guard conditions true? [6p]
9. Consider the hybrid system in Exercise 11.1 with totally inelastic impacts. Try to model the Newton’s Cradle with with totally inelastic impacts using the automaton defined in Question (a) of Exercise 11.1 in MatLab/Simulink. Please enclose the important parts of your code or simulink models to motivate your modeling method. [4p]

10. Simulate the system. Using the plots of the state trajectory and the hybrid model, explain what happens during the simulation. When will transitions take place? When are the guard conditions true? [4p]
    Hint: Try to simulate with different time step and simulation time to see the evolving details.

11. Can you visualize the existence of Zeno behavior in your simulation, as analyzed in Question (b) of Exercise 11.1? Please justify your response. [4p]
Appendix

Question 6 is to model Newton’s cradle as an automaton, i.e., as a collection $H = (Q, X, f, \text{Init}, D, E, G, R)$. When doing so, please follow this notation.

- First define a vector of continuous states $x = (\ldots)^T$ and a number of discrete states $\{q_1, \ldots, q_N\}$. What is your continuous state space $X$ and discrete state space $Q$?

- Then define the continuous dynamics associated with each discrete state, i.e., write down a function $f(q_i, x)$ for each discrete state $q_i$, such that
  $$\dot{x} = f(q_i, x) = \left[ \ldots \right].$$

- When the simulation starts, what is the initial value of your discrete and continuous states? Write it on the form
  $$\text{Init} = q_i \times (x_1^0, \ldots, x_n^0).$$

- Then define your edges, i.e., all possible transitions between discrete states. As an example, a system with three states could have the edge set
  $$E = \{(q_1, q_2), (q_2, q_3), (q_3, q_2), (q_3, q_3)\};$$
  (Note that the edges are directed and that edges can lead from a state to itself.)

- For each discrete state $q_i$, define the domain $D(q_i)$. Each such domain is a subset of $X$, describing what continuous states are allowed. An example is $D(q_1) = \{x : x_1 > 0\}$
  Also define the guard conditions for each edge, i.e., a condition on $x$ for when that transition can take place. Continuing the example above, the transition from $q_1$ to $q_2$ may only be allowed if $x_1 < 1$. This can be written $G(q_1, q_2) = \{x : x_1 < 1\}$.

- Finally define the reset map for each edge. It is a function giving the new values $x^+$ of the continuous states when the system transitions from a discrete state to another. An example:
  $$x^+ = R(q_1, q_2, x) = \begin{cases} (0, \ldots, 0)^T & \text{if } x_1 < 0 \\ 2x & \text{if } x_1 \geq 0 \end{cases}$$