

VEKTORANALYS

Kursvecka 1

GRADIENTEN

Kapitel 1-3
Sidor 3-28

TARGET PROBLEM

A mosquito is flying around in the room.

How does she find us during the night?

The mosquito must know:

- (A) How the temperature $T(x,y,z)$ changes along the flying direction
- (B) In which direction she must fly to be in a warmer place as soon as possible

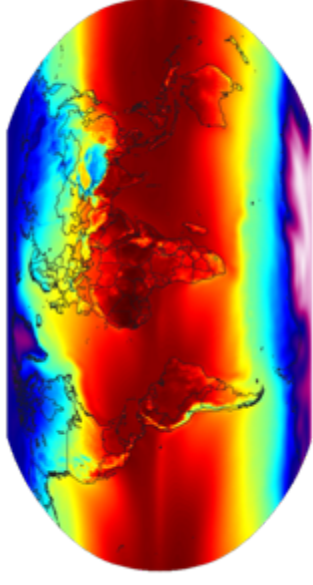
We need to:

- (1) introduce a **SCALAR FIELD**, $T(x,y,z)$
- (2) measure the **change of the scalar field** $T(x,y,z)$ in $\mathbb{R}^3$ (derivative)
- (3) find the **direction** for which **the change** of $T(x,y,z)$ **is maximum**

SCALAR FIELD AND VECTOR FIELD

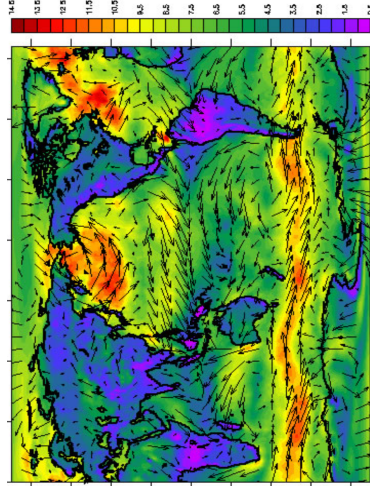
A **scalar field** associates a **real number** $\phi(x,y,z)$ **to each point** (x,y,z) of the space.

- Examples:
- temperature distribution in the space
 - pressure distribution in a fluid
 - potential around an electric charge



A **vector field** associates a **vector** $\vec{A}(x,y,z)$ **to each point** (x,y,z) of the space.

- Examples:
- velocity distribution in a fluid
 - magnetic field around a magnet
 - electric field around an electric charge



To solve our problem, today we will focus on scalar fields

LEVEL SURFACES

- Level surfaces are useful to visualize a scalar field.
- What is a level surface?

A **surface** on which the scalar field $\phi(x,y,z)$ is constant:

$$\phi(x,y,z)=c \quad (1)$$

- To create an “image” of the scalar field $\phi(x,y,z)$ we can consider a family of level surfaces:

$$\phi(x,y,z)=c+nh \quad (2)$$

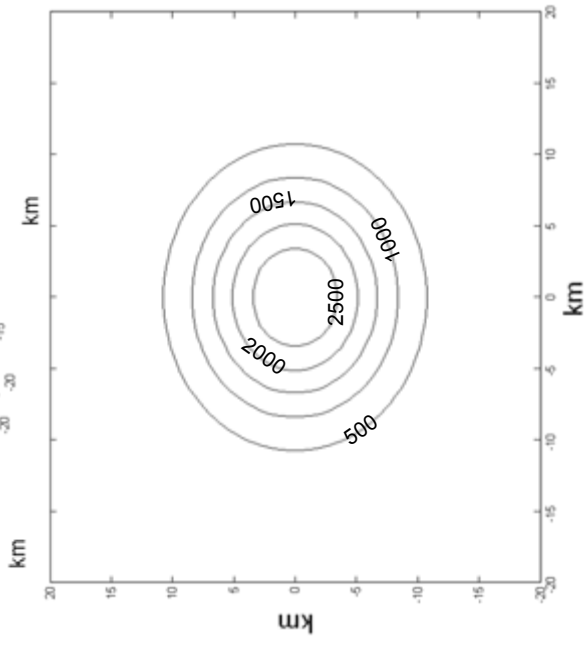
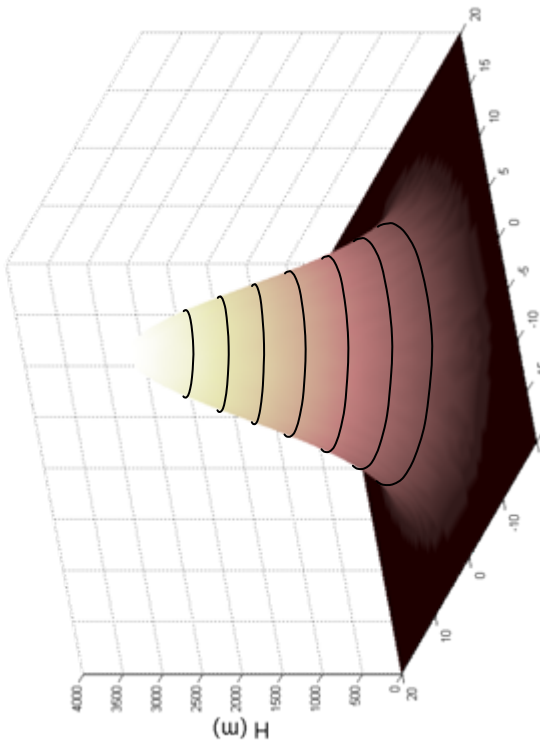
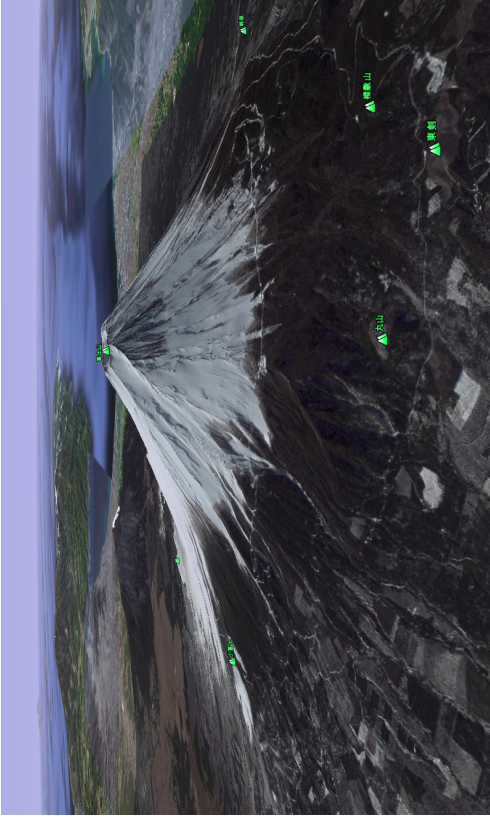
where h is a constant and $n=0, \pm 1, \pm 2, \pm 3, \dots$

To improve the details of the “image”, h must be reduced.

- In **two dimension** the scalar field is $\phi(x,y) \Rightarrow$ we have **level curves!**

EXAMPLE

- Assume that $H(x,y)$ is a scalar field corresponding to the height of a mountain



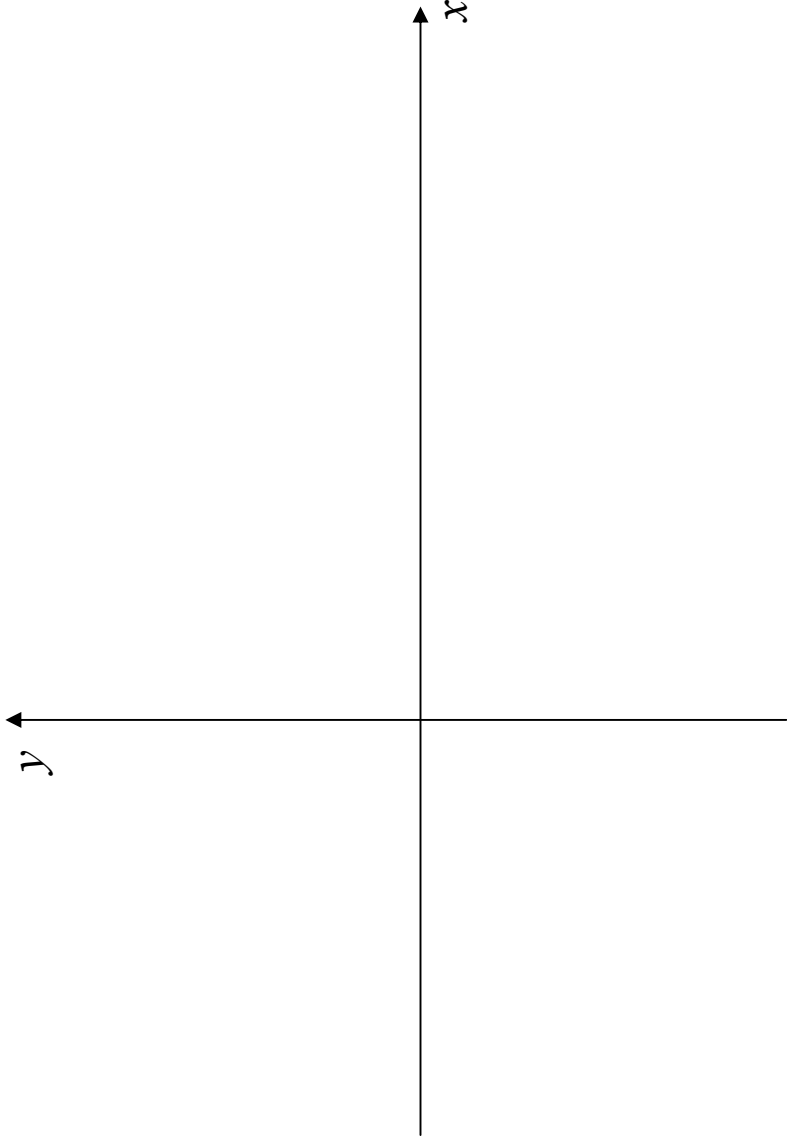
Let's consider a **family of level curves**:

$$H(x,y)=c+nh$$

EXERCISE

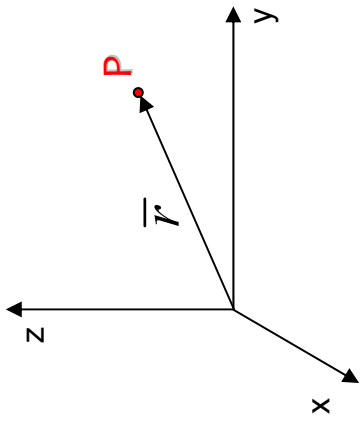
$$\phi = \frac{x^2}{4} + y^2$$

Plot the level curves of the scalar field:



POSITION VECTOR

- The **vector from the origin to the point $P=(x,y,z)$** is called **position vector $\vec{r}$**
- Note that $\vec{r}$ depends on the choice of the coordinate system
- $\vec{r}$ can be expressed with different notations:



$$\vec{r} = \vec{r}(x, y, z) \quad \vec{r} = (x, y, z) \quad \vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

- The **differential of a position vector** can be written as a **vector** whose components are the **differential of each position vector component**:

$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z \quad (3)$$



$$d\vec{r} = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z \quad (4)$$

THE GRADIENT

- Assume that $\phi(x,y,z)$ is a continuous and derivable scalar field

- DEFINITION:

$$\text{grad}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right)$$

(5)

the gradient of a scalar field is a vector field!

This definition assume a right-handed cartesian coordinate system.

EXERCISE: calculate the gradient of the vector field: $\phi = \frac{x^2}{4} + y^2$ and plot $\text{grad}\phi$ in the point $P=(2,0)$ and in the point $P=(0,-1)$

- Scalar field differential:

$$d\phi = \text{grad}\phi \cdot d\vec{r}$$

(6)

- Let's introduce :

- the amplitude of the position vector differential, ds , and
- the direction $\hat{e}$ ($\hat{e}$ is a unit vector, i.e. $|\hat{e}|=1$)

Equations (6) and (7) give:

$$d\vec{r} = \hat{e} ds$$

(7)

$$\frac{d\phi}{ds} = \text{grad}\phi \cdot \hat{e}$$

(8)

Directional derivative

The rate of variation of ϕ in a given direction corresponds to the component of the vector gradient in that direction

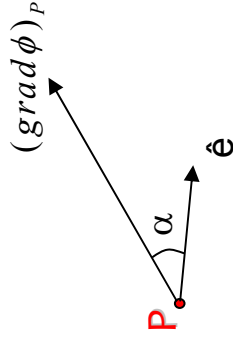
THE GRADIENT

THEOREM 1 (3.1 in the textbook)

The gradient in the point P is a vector that points to the direction in which the growth of ϕ in P is the highest.

The maximum increase of ϕ per unit length is $|(grad\phi)_P|$

PROOF



a- let's calculate the derivative in the direction $\hat{e}$ Eq. (8)

$$\frac{d\phi}{ds} = grad\phi \cdot \hat{e} = |grad\phi| \cos \alpha$$

b- this is maximum when:

$$\cos \alpha = 1$$

which implies:

$$\alpha=0 \quad (\hat{e} // grad\phi) \quad \text{and} \quad \frac{d\phi}{ds} = |grad\phi|$$

THE GRADIENT

THEOREM 2 (3.2 in the textbook)

The gradient in the point P is zero if ϕ has a maximum or a minimum in P

PROOF

From Equation (8): $\frac{d\phi}{ds} = \text{grad} \phi \cdot \hat{e}$

ϕ has a maximum or a minimum in P $\Rightarrow d\phi/ds=0$

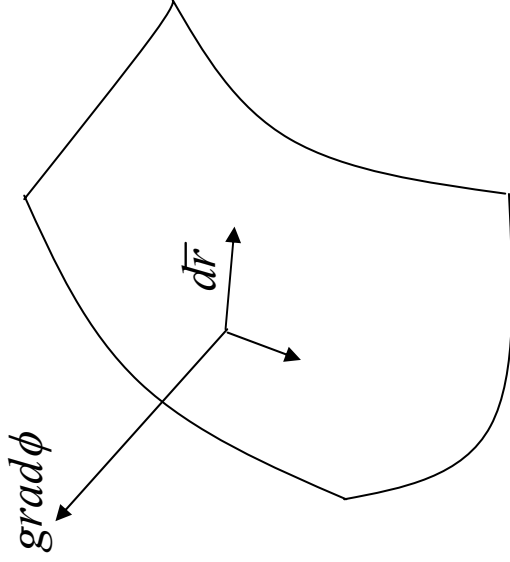
using equation (8), $d\phi/ds=0$ implies $\text{grad} \phi=0$

THE GRADIENT

THEOREM 3 (3.3 in the textbook)

The gradient of a scalar field $\phi(x,y,z)$ in the point P is orthogonal to the level surface $\phi=c$ in P.

PROOF



a- Let's do a small movement $d\vec{r}$ along the level surface

b- Remember that on the level surface ϕ is constant:

$$d\phi=0$$

c- Then, using equation (6):

$$d\phi = \text{grad}\phi \cdot d\vec{r} = 0$$

d- This implies that $\text{grad}\phi$ is perpendicular to $d\vec{r}$

e- $\text{grad}\phi$ is perpendicular to each $d\vec{r}$ on the level surface $\text{grad}\phi$ is perpendicular to the level surface

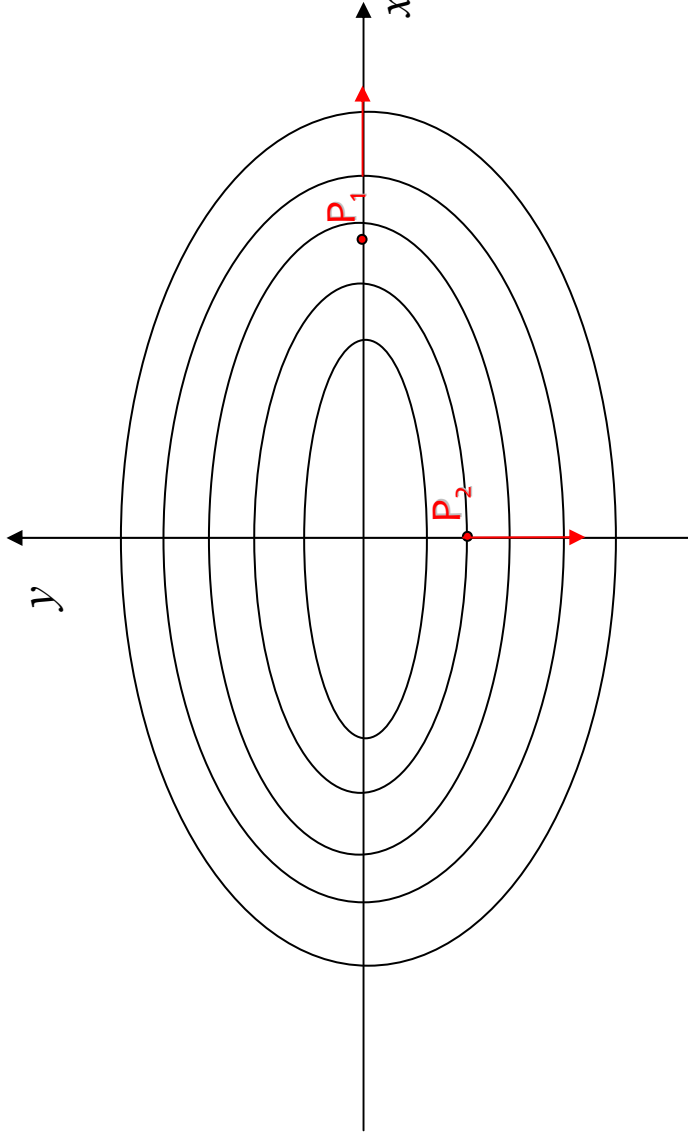
2D-EXAMPLE

- Theorems 1, 2 and 3 are valid also in two dimensions.
- $\text{grad}\phi$ is a vector field that:
 - in each point is orthogonal to the level curve in that point and
 - always points along the direction in which the height grows faster

Plot the level curves of the scalar field: $\phi = \frac{x^2}{4} + y^2$

$$\text{grad}\phi = \left(\frac{x}{2}, 2y \right)$$

and calculate the gradient in the points $P_1 = (2, 0)$ and $P_2 = (0, -1)$



$$\text{grad}\phi|_{P_1} = (1, 0)$$

$$\text{grad}\phi|_{P_2} = (0, -2)$$

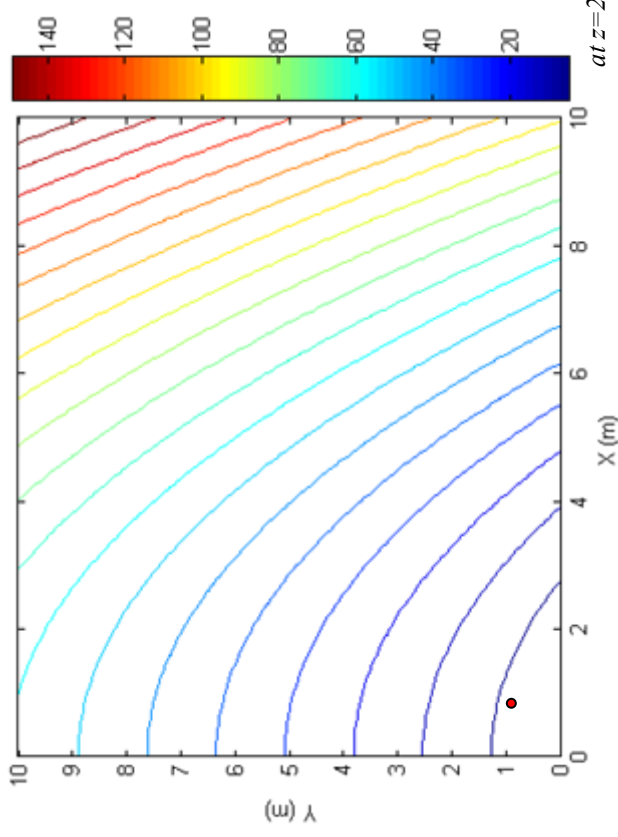
TARGET PROBLEM

A mosquito is flying around in the room.

The temperature is described by the scalar field:

$$T(x,y,z)=x^2+2yz-z \quad [^{\circ}\text{C}]$$

The mosquito is in the point $P=(1,1,2)$



- (a) In which direction the mosquito will fly to be in a warmer place as quick as possible?
- (b) How much the temperature changes in time if the mosquito flies with velocity 3m/s in direction $(-2,2,1)$?

TARGET PROBLEM

(a) In which direction the mosquito will fly to be warm as quick as possible?

We use **theorem 1**: The **gradient** in the point P is a **vector that points** to the direction in which the scalar field in P has the **highest growth**.

at $z=2$

From definition (5):

$$\text{grad}T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$$

$$T(x, y, z) = x^2 + 2yz - z$$

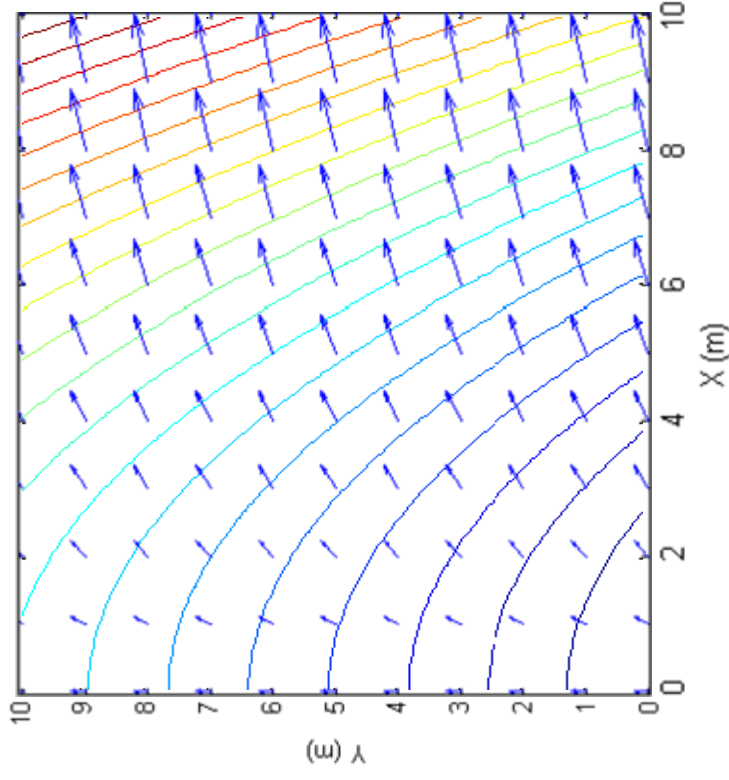
$$\frac{\partial T}{\partial x} = 2x, \quad \frac{\partial T}{\partial y} = 2z, \quad \frac{\partial T}{\partial z} = 2y - 1$$

$$\text{grad}T = (2x, 2z, 2y - 1)$$

The mosquito is in $P=(1, 1, 2)$

$$(\text{grad}T)_{P=(1,1,2)} = (2 \cdot 1, 2 \cdot 2, 2 \cdot 1 - 1) = (2, 4, 1)$$

The mosquito will fly in direction $(2, 4, 1)$



TARGET PROBLEM

(b) How fast the temperature changes if the mosquito flies with velocity 3m/s in direction (-2,2,1)?

We must calculate $\frac{dT}{dt}$ where t is the time

Using equation (6):

$$\frac{dT}{dt} \downarrow = \text{grad}T \cdot \frac{d\vec{r}}{dt} = \text{grad}T \cdot \hat{e} \frac{ds}{dt}$$

$$\text{where } \left\{ \begin{array}{l} \frac{ds}{dt} = |\vec{v}| = 3 \text{ m/s} \\ \hat{e} = \frac{\vec{v}}{|\vec{v}|} = \frac{(-2, 2, 1)}{|(-2, 2, 1)|} = \frac{(-2, 2, 1)}{\sqrt{(-2)^2 + 2^2 + 1^2}} = \frac{(-2, 2, 1)}{3} \end{array} \right\} \Rightarrow \frac{d\vec{r}}{dt} = \frac{(-2, 2, 1)}{3} \cdot 3 = (-2, 2, 1)$$

$$\frac{dT}{dt} = \text{grad}T \cdot \frac{d\vec{r}}{dt} = (2, 4, 1) \cdot (-2, 2, 1) = 5 \text{ [C/s]}$$

WHICH STATEMENT IS WRONG?

- 1- A scalar field associates a real number to a point in space (yellow)
- 2- A scalar field can be written as $\bar{A} = \bar{A}(x, y, z)$ (red)
- 3- If ϕ is a scalar then $grad\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right)$ in $\mathbb{R}^3$ (green)
- 4- The ϕ increase in a given direction depends on the directional derivative (blue)