

VEKTORANALYS

Kursvecka 2

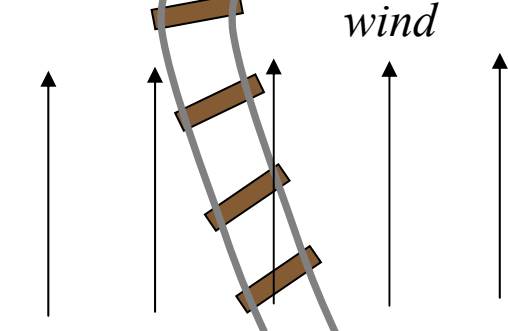
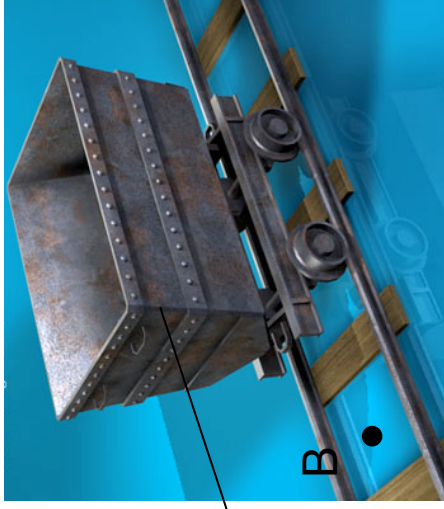
LINE INTEGRAL and FLUX INTEGRAL

Kapitel 4-5
Sidor 29-50

TARGET PROBLEM

We want to push a mine cart
along a path L from A to B.
But the wind is blowing.

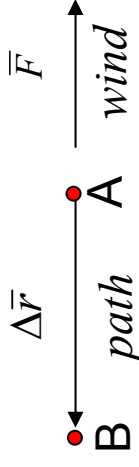
How much energy is needed?
(i.e. how much is the ‘work’?)



We will arrive to the final answer in three steps:

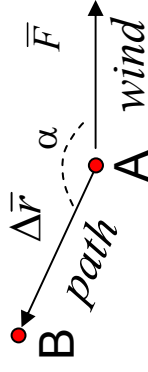
1- Path: straight line

Direction: the path L has the same direction of the wind



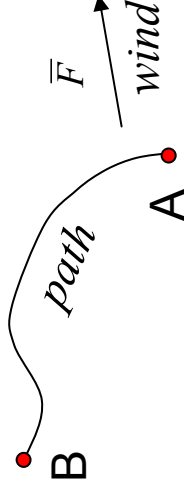
2- Path: straight line

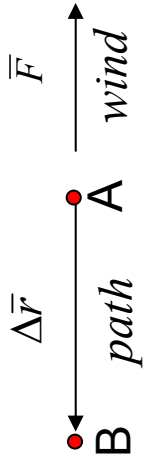
Direction: the path L has not the same direction of the wind



3- Path: curved line

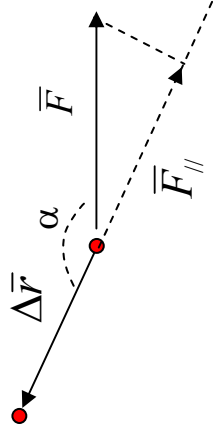
Wind not constant





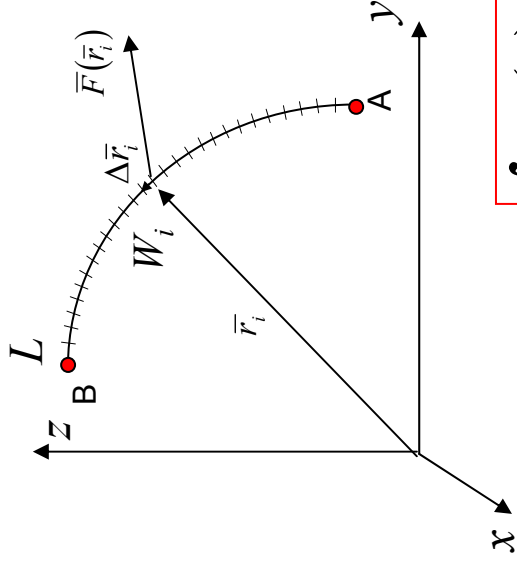
Step 1:

$$W = \left| \vec{F} \right| \left| \Delta \vec{r} \right|$$



Step 2:

$$W = \left| \vec{F}_{\parallel} \right| \left| \Delta \vec{r} \right| = \left| \vec{F} \right| \left| \Delta \vec{r} \right| \cos \alpha = \vec{F} \cdot \Delta \vec{r}$$



Step 3:

$$\left. \begin{aligned} W &= \sum_i W_i \\ W_i &\approx \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i \end{aligned} \right\} \Rightarrow W \approx \sum_i \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i$$

For "very small" segments:

$$W = \lim_{\Delta \vec{r}_i \rightarrow 0} \sum_i \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i \equiv \int_L \vec{F}(\vec{r}) \cdot d\vec{r}$$

$$\int_L \vec{F}(\vec{r}) \cdot d\vec{r}$$

is the line integral of $\vec{F}$ along the path L

We need to:

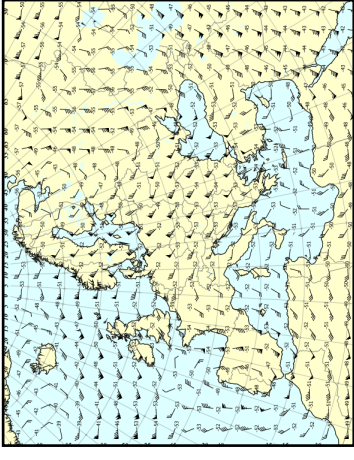
- introduce a **VECTOR FIELD**, $\vec{F}(\vec{r})$
- Define the **infinitesimal displacement** $d\vec{r}$ along the path L

VECTOR FIELD

A **vector field** associates a **vector** $\vec{A}(x,y,z)$ to **each point** (x,y,z) of the space.

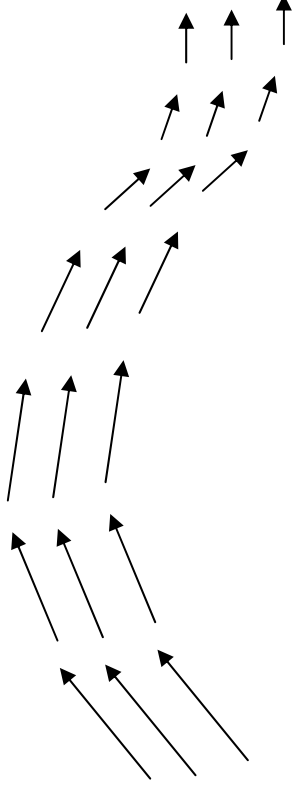
Examples:

- velocity distribution in a fluid
- magnetic field around a magnet
- electric field around an electric charge



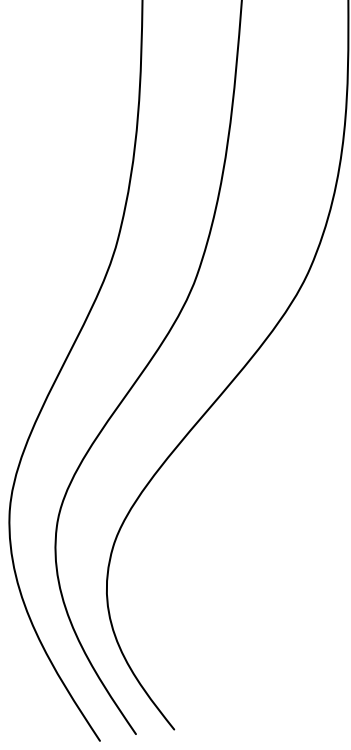
Two typical ways to represent a vector field:

1- Arrow field



- The arrow length is proportional to the field amplitude
- The arrow direction shows the field direction

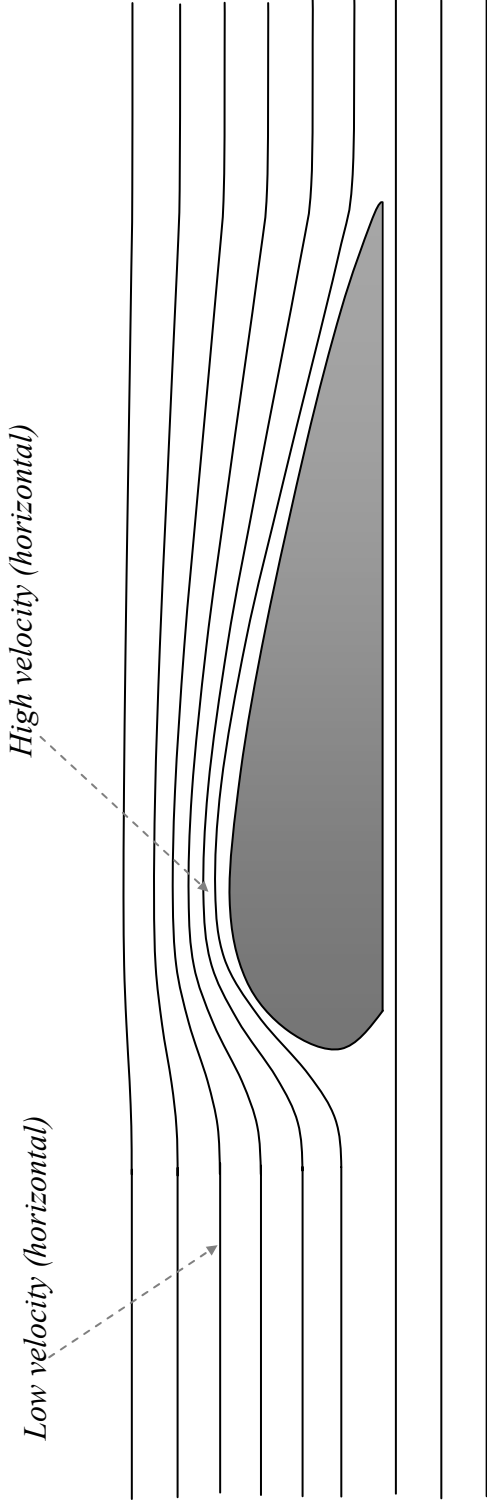
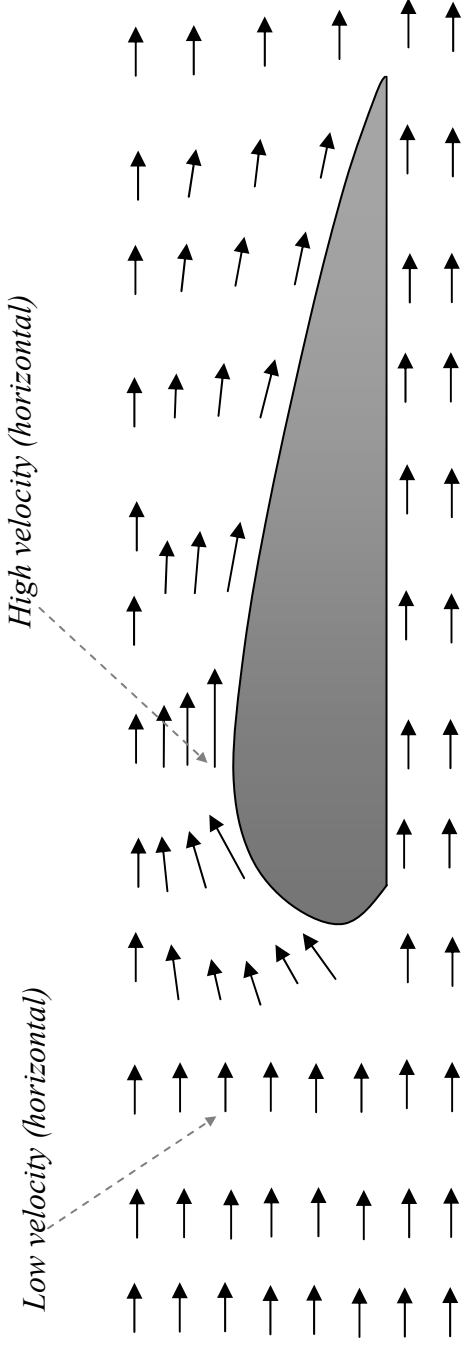
2- Line field



- The tangent to the curves is parallel to the vector field in all points.
- The density of the lines is proportional to the strength of the field.

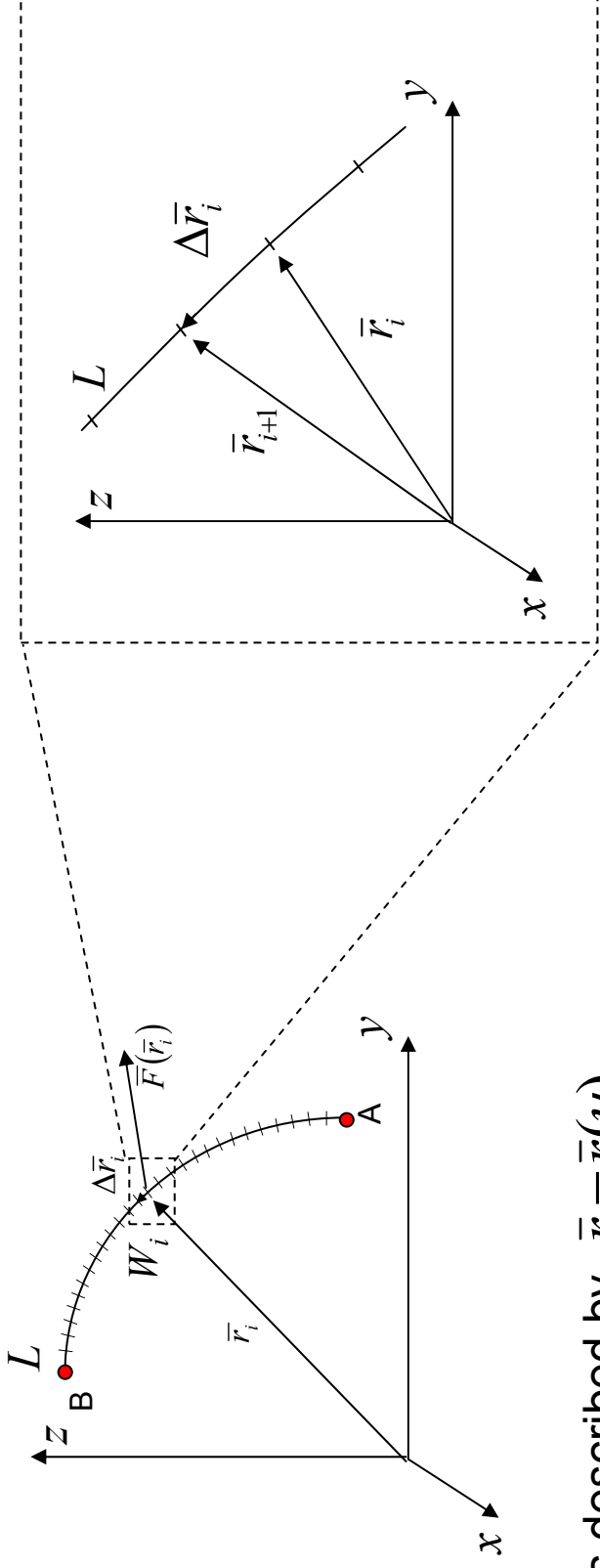
VECTOR FIELD

The airplane wing example (velocity field of air around a wing)



EXERCISE: 1- plot the vector field $\vec{F}(\vec{r}) = (x, 0)$
 2- write $\vec{F}(\vec{r})$ on the curve defined by: $\vec{r}(u) = (u, u^2)$

$d\vec{r}$ AND THE LINE INTEGRAL



L is described by $\vec{r} = \vec{r}(u)$

$$\left. \begin{aligned} \vec{r}_i &= \vec{r}(u_i) \\ \vec{r}_{i+1} &= \vec{r}(u_{i+1}) \\ u_{i+1} &= u_i + \Delta u \end{aligned} \right\} \Rightarrow d\vec{r} = \lim_{\Delta u \rightarrow 0} \Delta \vec{r}_i = \lim_{\Delta u \rightarrow 0} [\vec{r}(u_{i+1}) - \vec{r}(u_i)] = \lim_{\Delta u \rightarrow 0} \left[\frac{\vec{r}(u_{i+1}) - \vec{r}(u_i)}{\Delta u} \right] \Delta u = \frac{d\vec{r}}{du} du$$

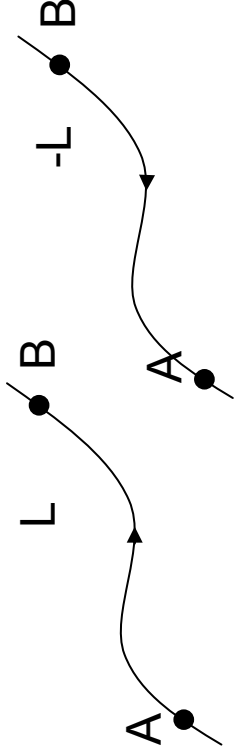
So, the line integral can be calculated as:

$$\int_L \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(u)) \cdot \frac{d\vec{r}}{du} du$$

EXERCISE: Calculate $\int_L \vec{F}(\vec{r}) \cdot d\vec{r}$ with $\vec{F}(\vec{r}) = (x, 0)$ and L defined by $\vec{r}(u) = (u, u^2)$
 $u : 0 \rightarrow 1$

LINE INTEGRAL (some useful properties)

THEOREM 1 (4.2 in the textbook)



(3)

$$\int_{-L} \vec{F}(\vec{r}) \cdot d\vec{r} = - \int_L \vec{F}(\vec{r}) \cdot d\vec{r}$$

PROOF

If all line elements change sign then also the integral will change sign.

DEFINITION

The line of integral of $\vec{A}$ along a closed curve C is called **circulation of $\vec{A}$ along C**:

$$\oint_C \vec{A}(\vec{r}) \cdot d\vec{r}$$

(4)

LINE INTEGRAL

DEFINITION: A vector field $\vec{A}$ is called conservative if: $\oint_C \vec{A}(\vec{r}) \cdot d\vec{r} = 0$

THEOREM 2 (4.3 in the textbook)

The circulation of $\vec{A}$ along a close curve C is zero if and only if for all points P and Q the line integral of A from P to Q is independent from the integration path between P and Q.

PROOF

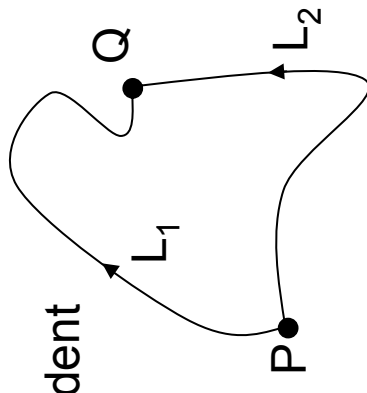
Assume that L_1 and L_2 are two curves from P to Q.

Then $L=L_1-L_2$ is a closed curve.

(1) The circulation is zero $\Rightarrow$ the line integral from P to Q is independent from the path

$$\left. \begin{aligned} \int_L \vec{A}(\vec{r}) \cdot d\vec{r} &= 0 \\ \oint_L \vec{A}(\vec{r}) \cdot d\vec{r} &= \int_{L_1-L_2} \vec{A}(\vec{r}) \cdot d\vec{r} = \int_{L_1} \vec{A}(\vec{r}) \cdot d\vec{r} - \int_{L_2} \vec{A}(\vec{r}) \cdot d\vec{r} \end{aligned} \right\} \Rightarrow \int_{L_1} \vec{A}(\vec{r}) \cdot d\vec{r} = \int_{L_2} \vec{A}(\vec{r}) \cdot d\vec{r}$$

The line integral is independent from the integration path!



(2) The line integral from P to Q is independent from the path $\Rightarrow$ the circulation is zero.

$$\left. \begin{aligned} \int_{L_1} \vec{A}(\vec{r}) \cdot d\vec{r} &= \int_{L_2} \vec{A}(\vec{r}) \cdot d\vec{r} \\ \oint_L \vec{A}(\vec{r}) \cdot d\vec{r} &= \int_{L_1-L_2} \vec{A}(\vec{r}) \cdot d\vec{r} = \int_{L_1} \vec{A}(\vec{r}) \cdot d\vec{r} - \int_{L_2} \vec{A}(\vec{r}) \cdot d\vec{r} \end{aligned} \right\} \Rightarrow \oint_L \vec{A}(\vec{r}) \cdot d\vec{r} = 0$$

The circulation is zero

LINE INTEGRAL

THEOREM 3 (4.4 in the textbook)

If $\vec{A} = \text{grad} \phi$:

$$\int_P^Q \vec{A}(r) \cdot d\vec{r} = \phi(Q) - \phi(P) \quad (5)$$

So the line integral is independent from the integration path and depends only on the starting point and on the ending point

PROOF

If $\vec{r}(u)$ is a curve from P to Q then, using the chain rule for the partial derivative:

$$\begin{aligned} \int_{\vec{r}(u)} \vec{A}(\vec{r}) \cdot d\vec{r} &= \int_P^Q \text{grad} \phi \cdot \frac{d\vec{r}}{du} du = \int_P^Q \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \cdot \left(\frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du} \right) du = \\ &= \int_P^Q \left(\frac{\partial \phi}{\partial x} \frac{dx}{du} + \frac{\partial \phi}{\partial y} \frac{dy}{du} + \frac{\partial \phi}{\partial z} \frac{dz}{du} \right) du = \int_P^Q \frac{d}{du} \phi(\vec{r}(u)) du = \phi(Q) - \phi(P) \end{aligned}$$

Or, easier: $\int_L \vec{A}(\vec{r}) \cdot d\vec{r} = \int_L \text{grad} \phi \cdot d\vec{r} = \int_L d\phi = \phi(B) - \phi(A)$

OTHER KINDS OF LINE INTEGRALS

- It is possible to combine scalar and vector line elements in many different ways along a curve L and thus get different kinds of line integrals

- Some examples:
$$\int_L \phi(\vec{r}) ds \quad (6)$$

$$\int_L \phi(\vec{r}) d\vec{r} \quad (7)$$

where $d\vec{r} = \hat{e} ds$

$$\int_L A(\vec{r}) \times d\vec{r} \quad (8)$$

- To calculate the integrals:

$$\vec{r} \rightarrow \vec{r}(u)$$

$$L \rightarrow [a, b] \quad \text{where } u: a \rightarrow b$$

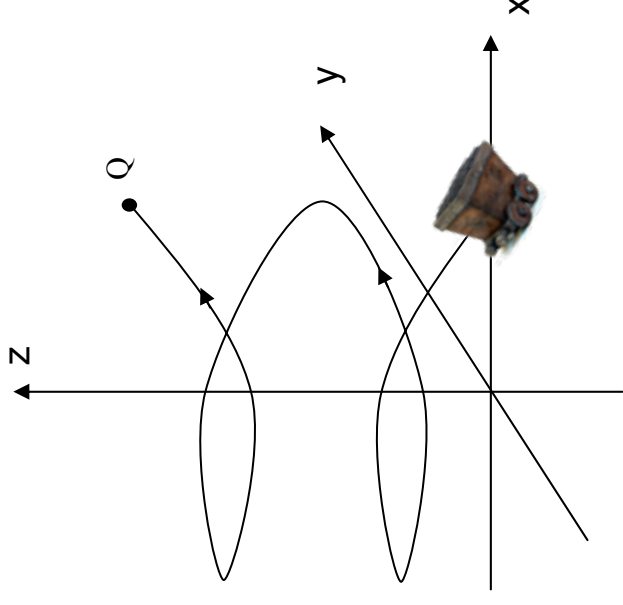
$$d\vec{r} = \frac{d\vec{r}}{du} du \quad \text{or} \quad ds = \left| \frac{d\vec{r}}{du} \right| du$$

EXAMPLE

The force is: $\vec{F} = (-yz, xz, -1)$

The path is $L: \vec{r} = (\cos u, \sin u, u)$ with $u: 0 \rightarrow 4\pi$

Calculate the work from $P=(1,0,0)$ till $Q=(1,0,4\pi)$



from definition (2)

$$W = \int_L \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{a=0}^{b=4\pi} \vec{F}(\vec{r}(u)) \frac{d\vec{r}}{du} du$$

$$\left. \begin{aligned} \vec{F}(\vec{r}(u)) &= (-(\sin u)u, (\cos u)u, -1) \\ \frac{d\vec{r}}{du} &= (-\sin u, \cos u, 1) \end{aligned} \right\} \Rightarrow \vec{F}(\vec{r}(u)) \cdot \frac{d\vec{r}}{du} = (u \sin^2 u + u \cos^2 u - 1) = u(\sin^2 u + \cos^2 u) - 1 = u - 1$$

$$W = \int_L \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^{4\pi} (u-1) du = \left[\frac{u^2}{2} - u \right]_0^{4\pi} = 8\pi^2 - 4\pi$$

WHICH STATEMENT IS WRONG?

- 1- The image area of a vector field $\vec{A}$ is composed of vectors (yellow)
- 2- The line integral $\int \vec{F} \cdot d\vec{r}$ is a scalar (red)
- 3- The sign of the line integral $\int \vec{F} \cdot d\vec{r}$ depends on the integration path (green)
- 4- The gradient of a vector field can be written as: $\text{grad } \vec{A}$ (blue)

TARGET PROBLEM

We are making cranberry juice.
After cranberries are squeezed,
It is better to filter the juice!

How much juice flows through the cloth each second?

We need :

- (1) to understand how to calculate the **flux**
of a **VECTOR FIELD** $\vec{v}(x, y, z)$
- (2) a method to integrate the flux over the whole surface.



THE FLUX

In the juice example, the flux F is the volume of the fluid ΔV that flows through the surface in the time Δt . $F = \frac{\Delta V}{\Delta t}$

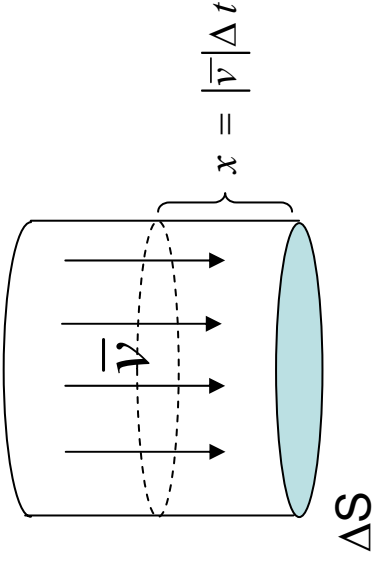
STEP 1: - the fluid velocity is perpendicular

to the surface

- the surface is not curved

$$\Delta V = x \Delta S = |\bar{v}| \Delta t \Delta S$$

$$F = \frac{\Delta V}{\Delta t} = |\bar{v}| \Delta S$$

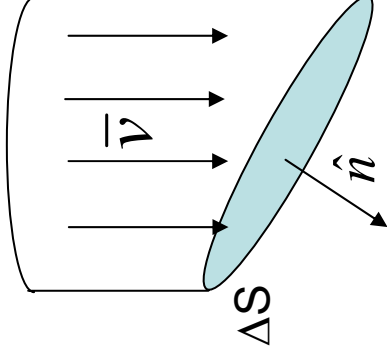


STEP 2: - the fluid velocity is NOT

perpendicular to the surface

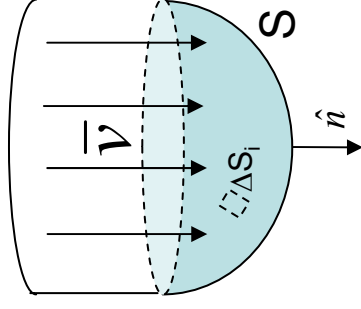
- the surface is not curved

$$F = |\bar{v}_\perp| \Delta S = \bar{v} \cdot \hat{n} \Delta S = \bar{v} \cdot \Delta \vec{S}$$



STEP 3: - the surface is curved

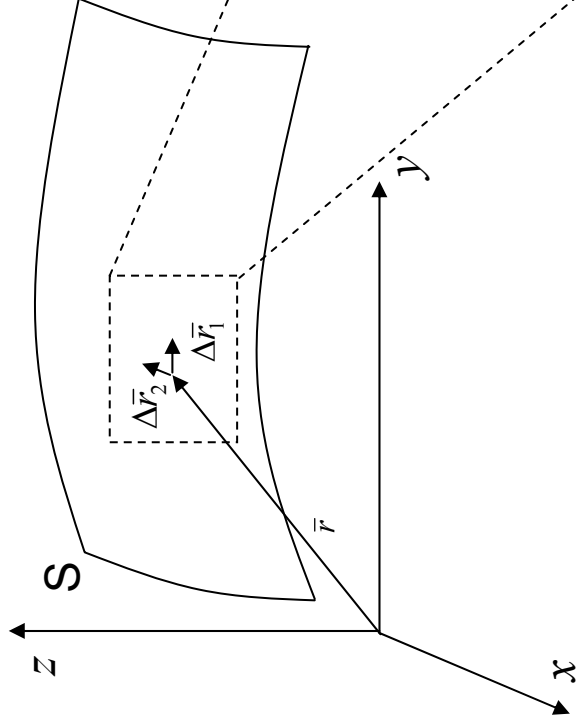
$$F = \sum_i F_i = \lim_{\Delta S_i \rightarrow 0} \sum_i \bar{v}_i \cdot \Delta \vec{S}_i \equiv \int_S \bar{v} \cdot d\vec{S}$$



$\int_S \bar{v} \cdot d\vec{S}$ is the flux integral of $\bar{v}$ on the surface S

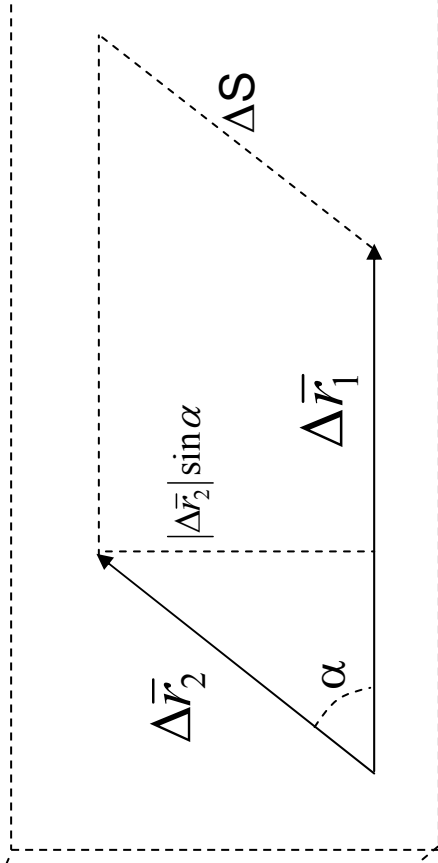
$d\vec{S}$ AND FLUX INTEGRAL

- Assume that the surface S is parameterized by $\vec{r} = \vec{r}(u, v)$



Let's consider two displacements,

- due a change in u : $\vec{r}_1 = \vec{r}(u + \Delta u, v) \Rightarrow \Delta \vec{r}_1 = \vec{r}_1 - \vec{r}$
- due a change in v : $\vec{r}_2 = \vec{r}(u, v + \Delta v) \Rightarrow \Delta \vec{r}_2 = \vec{r}_2 - \vec{r}$



The area ΔS is $\Delta S = |\Delta \vec{r}_2| \sin \alpha |\Delta \vec{r}_1| = |\Delta \vec{r}_1 \times \Delta \vec{r}_2|$

$\hat{n}$ is perpendicular to S . But also $\Delta \vec{r}_1 \times \Delta \vec{r}_2$ is perpendicular to S

$$\Rightarrow \Delta \vec{S} = \hat{n} \Delta S = \Delta \vec{r}_1 \times \Delta \vec{r}_2$$

$d\bar{S}$ AND FLUX INTEGRAL

$$d\bar{S} = \lim_{\substack{\Delta u \rightarrow 0 \\ \Delta v \rightarrow 0}} \Delta\bar{S} = \lim_{\substack{\Delta u \rightarrow 0 \\ \Delta v \rightarrow 0}} \Delta\bar{r}_1 \times \Delta\bar{r}_2$$

$$\begin{aligned} d\bar{r}_1 &= \lim_{\Delta u \rightarrow 0} \Delta\bar{r}_1 = \lim_{\Delta u \rightarrow 0} \bar{r}(u + \Delta u, v) - \bar{r}(u, v) = \\ &= \lim_{\Delta u \rightarrow 0} \frac{\bar{r}(u + \Delta u, v) - \bar{r}(u, v)}{\Delta u} \Delta u = \frac{\partial \bar{r}(u, v)}{\partial u} du \end{aligned}$$

$$\Rightarrow d\bar{S} = \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} du dv$$

in the same way:

$$d\bar{r}_2 = \frac{\partial \bar{r}(u, v)}{\partial v} dv$$

So, the flux integral of the vector field $\bar{v}$ on the surface $\bar{S}$ can be calculated as:

$$\int_S \bar{v} \cdot d\bar{S} = \int_{u,v} \bar{v}(\bar{r}(u, v)) \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right) du dv$$

EXAMPLE

Calculate the flux of the vector field $\vec{A} = (xy, 0, z^2)$

through the surface S: $z = x^2 + y^2$

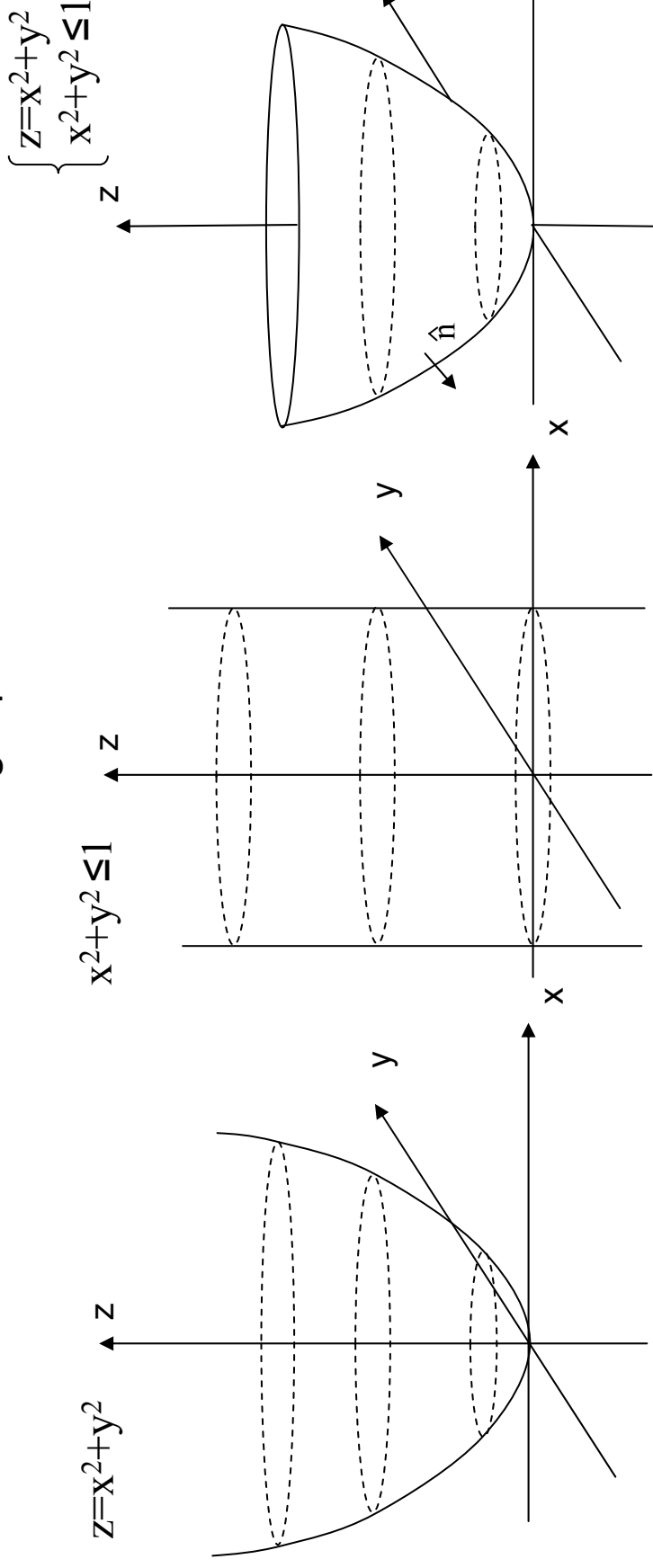
*This defines in which direction we will calculate the flux.
 $\hat{n}$ is chosen so that z-component is negative.*

$$\hat{n} \cdot \hat{e}_z < 0$$

SOLUTION: 1- figure

2- Parameterization of $\vec{S}$

3- Flux calculation using equation 10



EXAMPLE

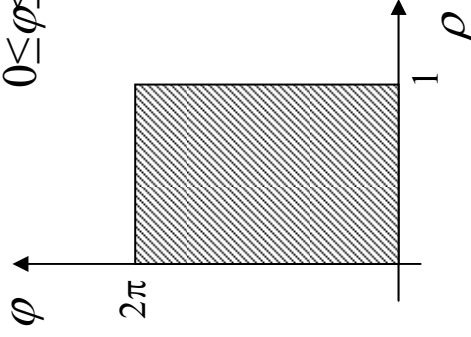
Parameterization of $\bar{S}$ $z=x^2+y^2$
 $x^2+y^2 \leq 1$

$$\bar{r}(\rho, \varphi) \begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \\ z = x^2 + y^2 = (\rho \sin \varphi)^2 + (\rho \sin \varphi)^2 = \rho^2 \end{cases}$$

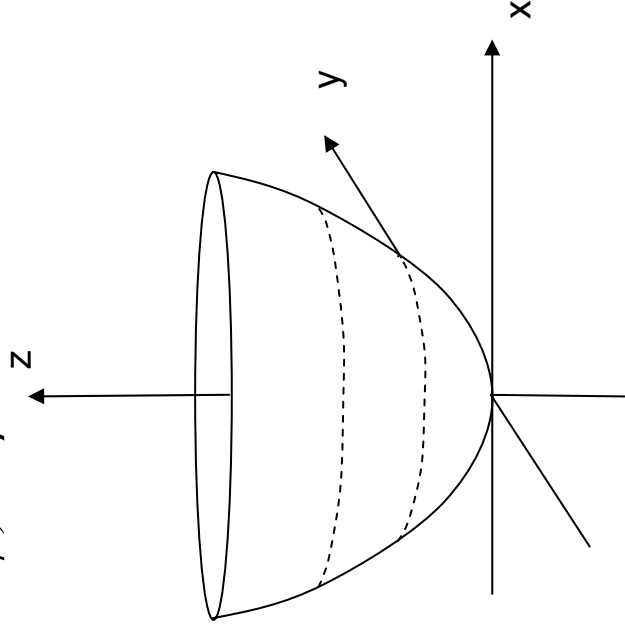
ρ and φ are polar coordinates

$$0 \leq \rho \leq 1$$

$$0 \leq \varphi \leq 2\pi$$



$$\bar{r}(\rho, \varphi)$$



Parameterization of the vector field: $\bar{A} = (xy, 0, z^2) = (\rho^2 \sin \varphi \cos \varphi, 0, \rho^4)$

EXAMPLE

Flux calculation using equation 10

$$\iint_{\bar{S}} \bar{A} \cdot d\bar{S} = \iint_{\bar{S}} \bar{A}(\bar{r}(\rho, \varphi)) \cdot \left(\frac{\partial \bar{r}}{\partial \rho} \times \frac{\partial \bar{r}}{\partial \varphi} \right) d\rho d\varphi$$

$$\left. \begin{aligned} \frac{\partial \bar{r}}{\partial \rho} &= (\cos \varphi, \sin \varphi, 2\rho) \\ \frac{\partial \bar{r}}{\partial \varphi} &= (-\rho \sin \varphi, \rho \cos \varphi, 0) \end{aligned} \right\} \Rightarrow \left(\frac{\partial \bar{r}}{\partial \rho} \times \frac{\partial \bar{r}}{\partial \varphi} \right) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \cos \varphi & \sin \varphi & 2\rho \\ -\rho \sin \varphi & \rho \cos \varphi & 0 \end{vmatrix} = \begin{pmatrix} -2\rho^2 \cos \varphi, -2\rho^2 \sin \varphi, \rho \cos^2 \varphi + \rho \sin^2 \varphi \end{pmatrix} = \begin{pmatrix} -2\rho^2 \cos \varphi, -2\rho^2 \sin \varphi, \rho \end{pmatrix}$$

Note that $\left(\frac{\partial \bar{r}}{\partial \rho} \times \frac{\partial \bar{r}}{\partial \varphi} \right)$ has a positive z-component, while the flux was in the other direction.

So we ordinary solve the integral but then we change the sign in the answer!

EXAMPLE

$$\begin{aligned}
 \iint_{-S} \bar{A} \cdot d\bar{S} &= \iint_{-S} \bar{A}(\bar{r}(\rho, \varphi)) \cdot \left(\frac{\partial \bar{r}}{\partial \rho} \times \frac{\partial \bar{r}}{\partial \varphi} \right) d\rho d\varphi = \\
 &\int_0^{2\pi} \int_0^1 \left(\rho^2 \sin \varphi \cos \varphi, 0, \rho^4 \right) \cdot \left(-2\rho^2 \cos \varphi, -2\rho^2 \sin \varphi, \rho \right) d\rho d\varphi = \\
 &\int_0^{2\pi} \int_0^1 \left(-2\rho^4 \sin \varphi \cos^2 \varphi + 0 + \rho^5 \right) d\rho d\varphi = \\
 &\int_0^{2\pi} \left[-\frac{2}{5} \rho^5 \sin \varphi \cos^2 \varphi + \frac{1}{6} \rho^6 \right]_0^1 d\varphi = \int_0^{2\pi} \left(-\frac{2}{5} \sin \varphi \cos^2 \varphi + \frac{1}{6} \right) d\varphi = \\
 &\left[-\frac{2}{5} \left(-\frac{\cos^3 \varphi}{3} \right) + \frac{1}{6} \varphi \right]_0^{2\pi} = \frac{\pi}{3}
 \end{aligned}$$

But we must change sign! The answer is $-\frac{\pi}{3}$

WHICH STATEMENT IS WRONG?

- 1- The flux integral is a scalar (yellow)
- 2- Flux integrals can be calculated also on a closed surface. (red)
- 3- The perpendicular to the integration surface points out from an arbitrary side. (green)
- 4- The flux through a membrane can be calculated with flux integrals. (blue)