EP2200 Queuing theory and teletraffic systems

2nd lecture

Poisson process
Markov process

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Course outline

- Stochastic processes behind queuing theory (L2-L3)
  - Poisson process
  - Markov Chains
    - Continuous time
    - Discrete time
  - Continuous time Markov Chains and queuing Systems
- Markovian queuing systems (L4-L7)
- Non-Markovian queuing systems (L8-L10)
- Queuing networks (L11)

Outline for today

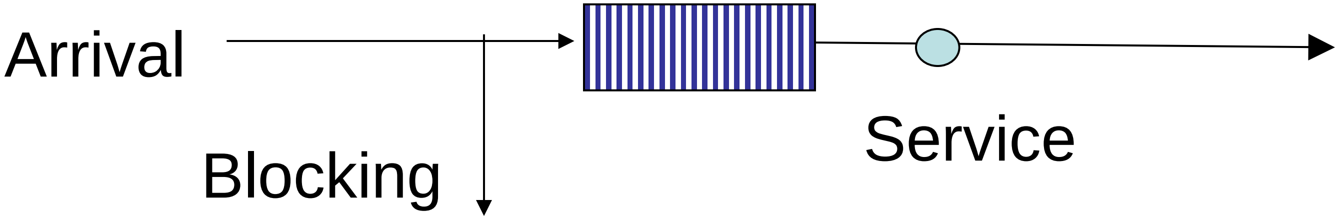
- Recall: queuing systems, stochastic process
- Poisson process – to describe arrivals and services
  - –properties of Poisson process
- Markov processes – to describe queuing systems
  - –continuous-time Markov-chains
- Graph and matrix representation

Recall from previous lecture

- Queuing theory: performance evaluation of resource sharing systems
- Specifically, for teletraffic systems

- Definition of queuing systems
- Performance triangle: service demand, server capacity and performance

- Service demand is random in time → theory of stochastic processes



Stochastic process

- Stochastic process
  - A system that evolves – changes its *state* - in *time* in a random way
  - Random variables indexed by a time parameter
    - continuous or discrete space
    - continuous or discrete time
  - State probability distribution
    - time dependent state probability distribution – ensemble average (probability density function, probability distribution function (or cumulative distribution function)

$$f_x(t) = P(X(t) = x), \; F_x(t) = P(X(t) \le x)$$

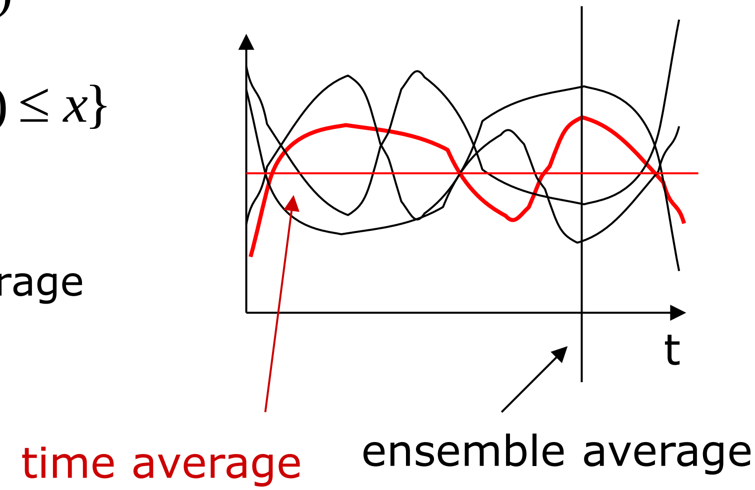
    - limiting state probability distribution

$$f_x = \lim_{t\to\infty} P(X(t) = x), \quad F_x = \lim_{t\to\infty} P\{X(t) \le x\}$$

    - stationary process

$$F_x(t+\tau) = F_x(t), \quad \forall t$$

    - ergodic process: ensemble average = time average



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- Transient and stationary state of the process

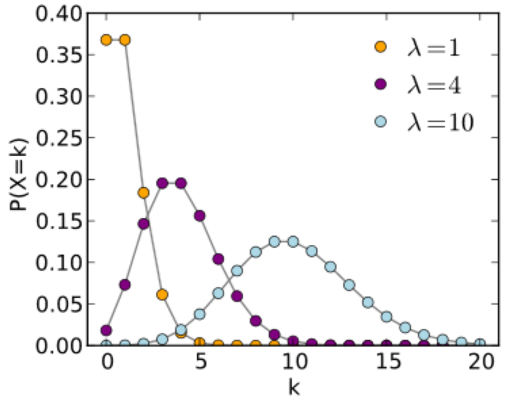
Poisson process

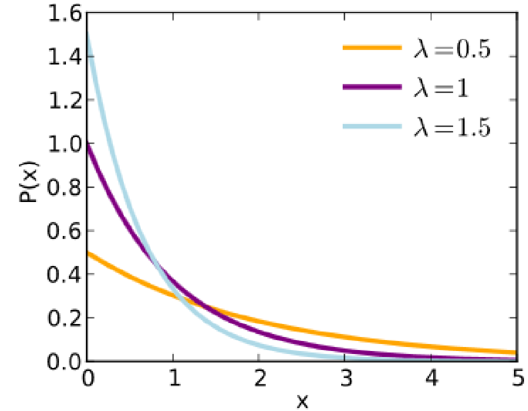
- Recall: key random variables and distributions
- Poisson distribution
  - Discrete probability distribution
  - Probability if a given number of events

$$P(X=k)=p_k=\frac{\lambda^k}{k!}e^{-\lambda}$$

- Exponential distribution
  - Continuous probability distribution

$$f(x)=p(x)=\lambda e^{-\lambda x}, \quad F(x)=P(X\le x)=1-e^{-\lambda x}$$





Wikipedia

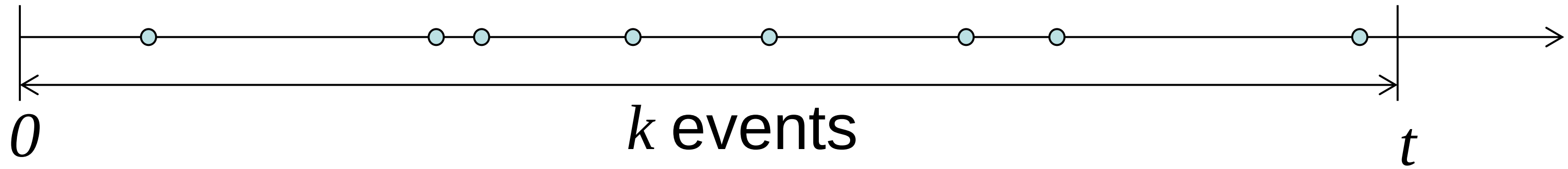
Poisson process

- Poisson process: to model arrivals and services in a queuing system
- Definition:
  - –Stochastic process – discrete state, continuous time
  - –X(t) : number of events (arrivals) in interval (0-t] (counting process)
  - –X(t) is Poisson distributed with parameter λt

$$P(X(t)=k)=p_k(t)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}, \quad E[X(t)]=\lambda t$$

  - –λ is called as the intensity of the Poisson process
  - –note, limiting state probabilities $p_k=\lim_{t\to\infty} p_k(t)$ do not exist

$p_k(t)$: Poisson distribution



Poisson process

- Def: The number of arrivals in period (0,t] has Poisson distribution with parameter λt, that is:

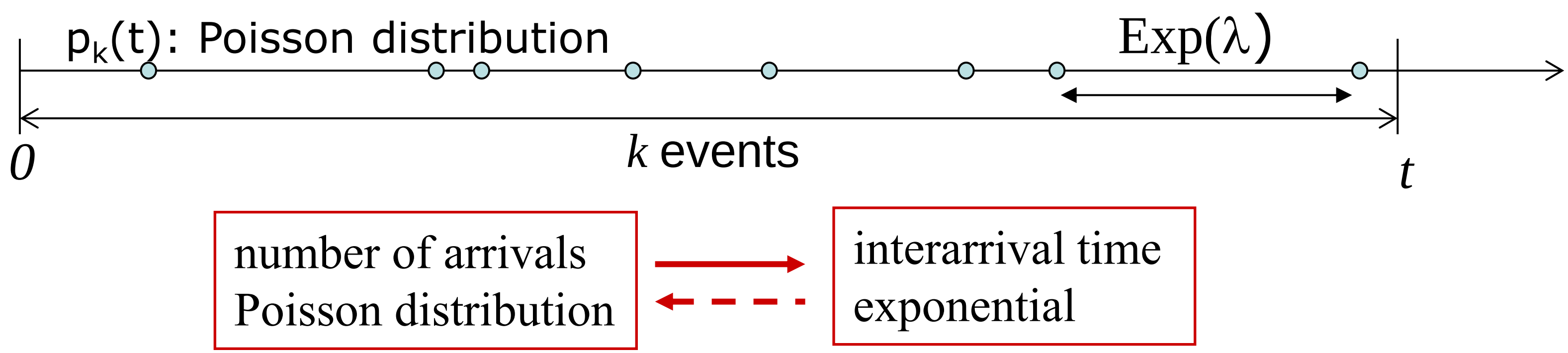
$$P(X(t)=k)=p_k(t)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

- Theorem: For a Poisson process, the time between arrivals (interarrival time) is exponentially distributed with parameter λ:
  - Recall exponential distribution:

$$f(t)=\lambda e^{-\lambda t}, \quad F(t)=P(\tau\le t)=1-e^{-\lambda t}, \quad E[\tau]=1/\lambda$$

  - Proof:

$$P(\tau<t)=P(\text{at least one arrival until } t)=1-P(\text{no arrival until } t)=1-e^{-\lambda t}$$



The memoryless property

- Def: a distribution is memoryless if:

$$P(\tau > t + s \mid \tau > s) = P(\tau > t)$$

- Example: the length of the phone calls
  - Assume the probability distribution of holding times (τ) is memoryless
  - Your phone calls last 30 minutes in average
  - You have been on the phone for 10 minutes already
  - What should we expect? For how long will you keep talking?

$$P(\tau > t + 10 \mid \tau > 10) = P(\tau > t)$$

  - It does not matter when you have started the call, if you have not finished yet, you will keep talking for another 30 minutes in average.

Exponential distribution and memoryless property

- Def: a distribution is memoryless if:

$$P(\tau > t+s \mid \tau > s) = P(\tau > t)$$

- Exponential distribution:

$$f(t) = \lambda e^{-\lambda t}, \quad F(t) = P(\tau \le t) = 1 - e^{-\lambda t}, \quad \bar{F}(t) = P(\tau > t) = e^{-\lambda t}$$

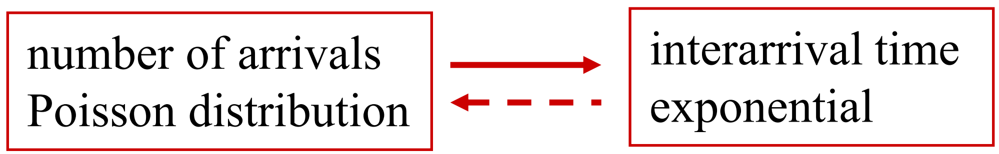
- The Exponential distribution is memoryless:

$$P(\tau > t+s \mid \tau > s) = \frac{P(\tau > t+s, \tau > s)}{P(\tau > s)} = \frac{P(\tau > t+s)}{P(\tau > s)} =$$

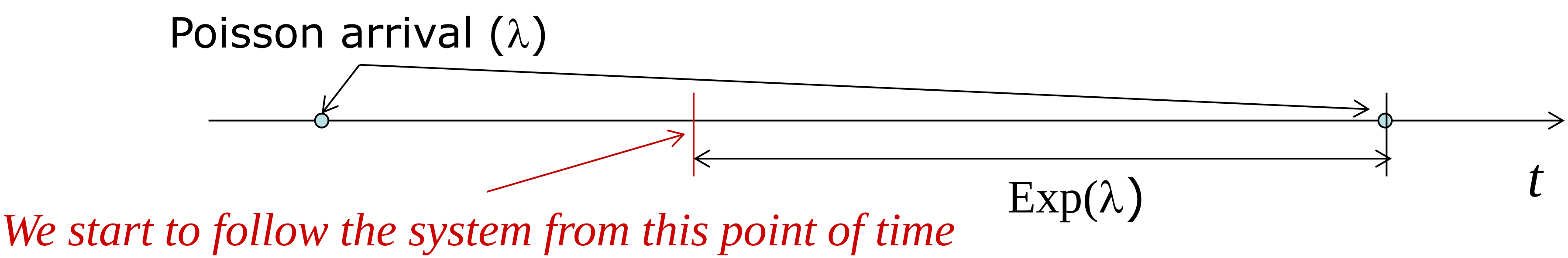
$$\frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = P(\tau > t)$$

Poisson process and exponential distribution

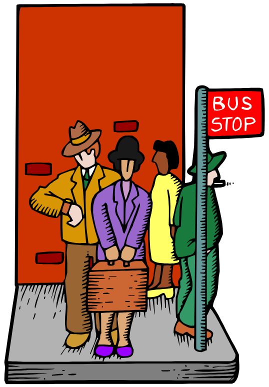
- Poisson arrival process implies exponential interarrival times
- Exponential distribution is memoryless



- For Poisson arrival process:
  the time until the next arrival does not depend on the time spent after the previous arrival



Group work



Waiting for the bus:

- Bus arrivals can be modeled as stochastic process
- The mean time between bus arrivals is 10 minutes. Each day you arrive to the bus stop at a random point of time. How long do you have to wait in average?

Consider the same problem, given that

a) Buses arrive with fixed time intervals of 10 minutes.

b) Buses arrive according to a Poisson process.

See "The hitchhiker's paradox" in Virtamo, Poisson process.

Properties of the Poisson process
(See also problem set 2)

1. The sum of Poisson processes is a Poisson process
   - The intensity is equal to the sum of the intensities of the summed (multiplexed, aggregated) processes
2. A random split of a Poisson process result in Poisson subprocesses
   - The intensity of subprocess i is λp_i, where p_i is the probability that an event becomes part of subprocess i
3. Poisson arrivals see time average (PASTA)
   - Sampling a stochastic process according to Poisson arrivals gives the state probability distribution of the process (even if the arrival changes the state)
   - Also known as ROP (Random Observer Property)
4. *Superposition of arbitrary renewal processes tends to a Poisson process (Palm theorem) – we do not prove*
   - Renewal process: independent, identically distributed (iid) inter-arrival times

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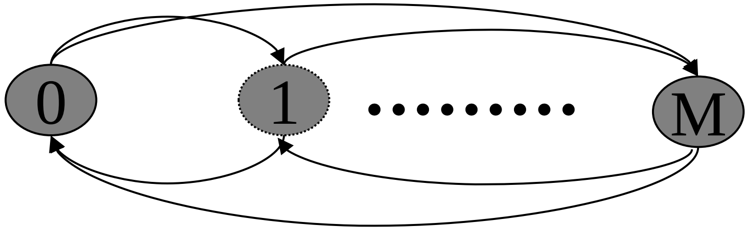
  - Continuous-time Markov-chains
  - Graph and matrix representation
  - Transient and stationary state of the process

Markov processes

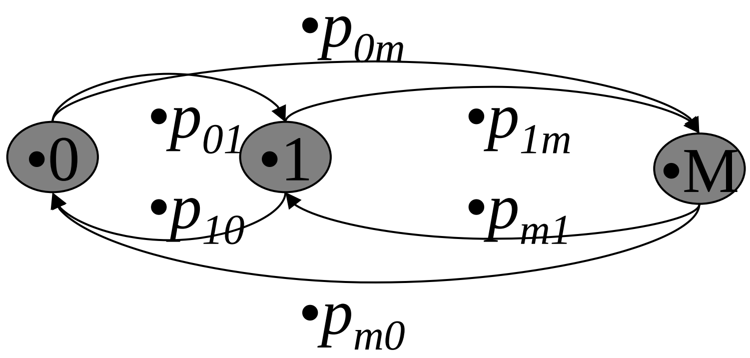
- Stochastic process
  - $p_i(t)=P(X(t)=i)$
- The process is a Markov process if *the future of the process depends on the current state only* - Markov property
  - $P(X(t_{n+1})=j \mid X(t_n)=i, X(t_{n-1})=l, \ldots, X(t_0)=m) = P(X(t_{n+1})=j \mid X(t_n)=i)$
  - Homogeneous Markov process: the probability of state change is unchanged by time shift, depends only on the time interval

    $P(X(t_{n+1})=j \mid X(t_n)=i) = p_{ij}(t_{n+1}-t_n)$
- Markov chain: if the state space is discrete
  - A homogeneous Markov chain can be represented by a graph:
    - States: nodes
    - State changes: edges



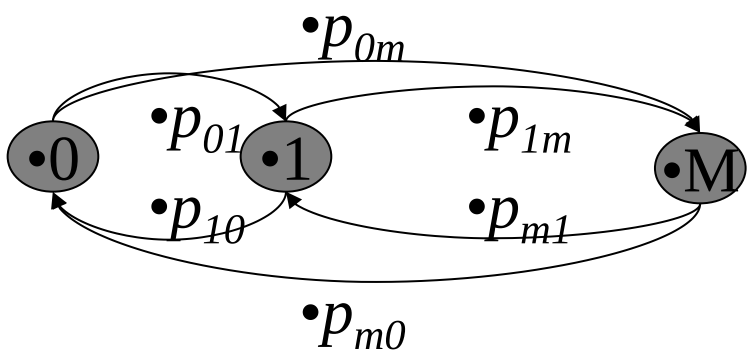
Discrete-time Markov-chains (detour)

- Discrete-time Markov-chain: the time is discrete as well
  - $X(0), X(1), \dots X(n), \dots$
  - Single step state transition probability for homogeneous MC: $P(X(n+1)=j \mid X(n)=i) = p_{ij}, \ \forall n$
- Example
  - Packet size from packet to packet
  - Number of correctly received bits in a packet
  - Queue length at packet departure instants …
    (get back to it at non-Markovian queues)



Discrete-Time Markov-chains

- Transition probability matrix:
  - The transitions probabilities can be represented in a matrix
  - Row *i* contains the probabilities to go from *i* to state *j=0, 1, …M*
    - P_{ii} *is the probability of staying in the same state*



$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0M} \\ p_{10} & p_{11} & & \vdots \\ \vdots & & \ddots & \vdots \\ p_{M0} & & \cdots & p_{MM} \end{bmatrix}, \quad \sum_j p_{ij} = 1 \quad \forall i$$

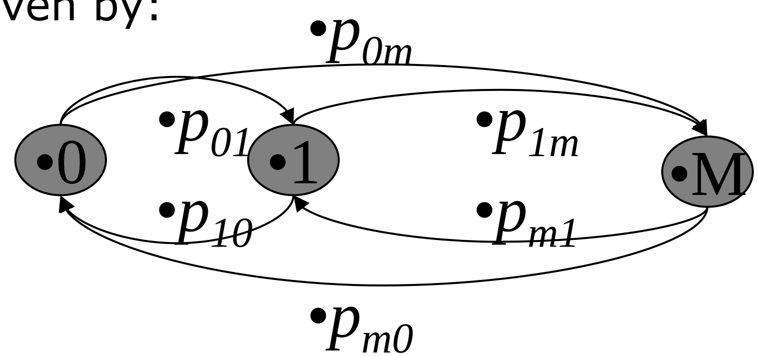
Discrete-Time Markov-chains

- The probability of finding the process in state *j* at time *n* is denoted by:
  - $p_j^{(n)} = P(X(n) = j)$
  - for all states and time points, we have:
    $$\underline{p}^{(n)} = \left[p_0^{(n)} \quad p_1^{(n)} \quad \cdots \quad p_M^{(n)} \right]$$

- The time-dependent (transient) solution is given by:

$$p_i^{(n+1)} = p_i p_{ii} + \sum_{j \neq i} p_j^{(n)} p_{ji}$$

$$\underline{p}^{(n+1)} = \underline{p}^{(n)} \mathbf{P} = \underline{p}^{(n-1)} \mathbf{P}\mathbf{P} = \cdots = \underline{p}^{(0)} \mathbf{P}^{n+1}$$



Discrete-Time Markov-chains

- Steady (or stationary) state exists if
  - The limiting probability vector exists
  - And is independent from the initial probability vector

$$\lim_{n\to\infty} \underline{p}^{(n)} = \underline{p} = \begin{bmatrix} p_0 & p_1 & \cdots & p_M \end{bmatrix}$$

- Stationary state probability distribution is give by:

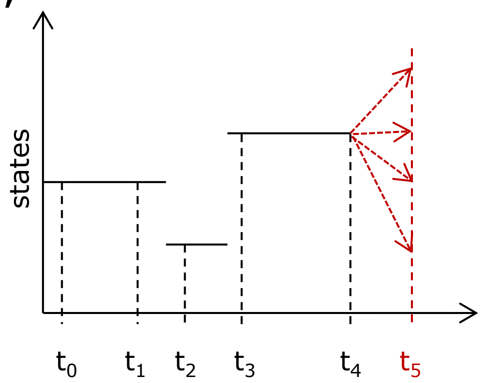
$$\underline{p} = \underline{p}\,\mathbf{P}, \quad \sum_{j=0}^{M} p_j = 1 \qquad \left(\underline{p}^{(n+1)} = \underline{p}^{(n)}\mathbf{P}\right)$$

- Note also:
  - The probability to remain in a state *j* for *m* time units has geometric distribution
    $p_{jj}^{m-1}\left(1 - p_{jj}\right)$
  - The geometric distribution is a memoryless discrete probability distribution (the only one)

Continuous-time Markov chains (homogeneous case)

- Continuous time, discrete space stochastic process, with Markov property, that is:

$$P(X(t_{n+1}) = j \mid X(t_n) = i, X(t_{n-1}) = l, \ldots X(t_0) = m) =$$
$$P(X(t_{n+1}) = j \mid X(t_n) = i), \quad t_0 < t_1 < \ldots < t_n < t_{n+1}$$



- State transition can happen in any point of time
- Example:
  - number of packets waiting at the output buffer of a router
  - number of customers waiting in a bank

- The time spent in a state has to be exponential to ensure Markov property:
  - the probability of moving from state *i* to state *j* sometime between t_n and t_{n+1} does not depend on the time the process already spent in state *i* before t_n.

Continuous-time Markov chains (homogeneous case)

- State change probability: $P(X(t_{n+1})=j \mid X(t_n)=i) = p_{ij}(t_{n+1}-t_n)$
- Characterize the Markov chain with the state transition rates instead:

$$q_{ij} = \lim_{\Delta t \to 0} \frac{P(X(t+\Delta t)=j|X(t)=i)}{\Delta t}, \quad i \neq j$$

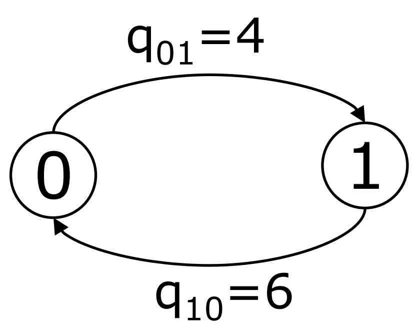
- rate (intensity) of state change

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$

- defined to easy calculation later on

- Transition rate matrix **Q**:

$$\mathbf{Q} = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0M} \\ \vdots & \ddots & & \\ & & & q_{(M-1)M} \\ q_{M0} & \cdots & q_{M(M-1)} & q_{MM} \end{bmatrix}$$



$$\mathbf{Q} = \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}$$

Summary

- Poisson process:
  - number of events in a time interval has Poisson distribution
  - time intervals between events has exponential distribution
  - The exponential distribution is memoryless
- Markov process:
  - stochastic process
  - future depends on the present state only, the Markov property
- Continuous-time Markov-chains (CTMC)
  - state transition intensity matrix
- Next lecture
  - CTMC transient and stationary solution
  - global and local balance equations
  - birth-death process and revisit Poisson process
  - Markov chains and queuing systems
  - discrete time Markov chains