Homework 3

Finite Differences and Absolute Stability

due 10/2-2014

Task 1: Finite Difference Scheme

Find the highest order approximation possible of the first derivative based on the grid values u_{i-1}, u_i, u_{i+1} and u_{i+2}. Assume equidistant grid spacing Δx.

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_i} \approx f(u_{i-1}, u_i, u_{i+1}, u_{i+2}) \tag{1}$$

a) Give the approximation for the derivative.

b) What is the leading error term? Of what order is the scheme?

c) Implement this derivative in a similar way as in Task 2a) of Homework 2 and assess numerically the order of its accuracy. Note that you do not need to compute truncation and round-off errors separately, just the global order of accuracy is required.

Task 2: Stability Criteria

The range of absolute stability of the Runge-Kutta 4^{th}-order method is studied. This method for an initial value problem of the form $\frac{\mathrm{d}u}{\mathrm{d}t} = f(u,t),\ u(t_0) = u_0$ is

$$u^{n+1} = u^n + \frac{\Delta t}{6}(f^n + 2k_1 + 2k_2 + k_3) \tag{2}$$

$$t^n = n\Delta t \tag{3}$$

where

$$f^n = f(u^n, t^n) \tag{4}$$

$$k_1 = f(u_1, t^{n+\frac{1}{2}}), \quad u_1 = u^n + \frac{\Delta t}{2} f^n, \quad t^{n+\frac{1}{2}} = t^n + \frac{\Delta t}{2} \tag{5}$$

$$k_2 = f(u_2, t^{n+\frac{1}{2}}), \quad u_2 = u^n + \frac{\Delta t}{2} k_1 \tag{6}$$

$$k_3 = f(u_3, t^{n+1}), \quad u_3 = u^n + \Delta t k_2 \tag{7}$$

Consider a simple linear test equation (*Dahlquist equation*):

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u\,. \tag{8}$$

Show that u^{n+1} can be written as a function of u^n and $z = \Delta t \lambda$ as follows

$$u^{n+1} = u^n\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right)\,. \tag{9}$$

The absolute stability criterion is

$$|G(z)| = \left|\frac{u^{n+1}}{u^n}\right| \leq 1\,. \tag{10}$$

Draw the region that corresponds to equation (10) in the complex z-plane.

Hint: The curve $|G(z)| = 1$ cuts the imaginary axis at ± 2.83.

Task 3: Modified Wavenumber

On an equidistant grid, the finite-difference derivative of a Fourier mode e^{ikx} can be found by multiplying the function value on each node with the so-called modified wavenumber $\tilde{k}(k)$.

To better understand that concept, consider a periodic function

$$f(x) : \mathbb{R} \to \mathbb{C}, f(x + 2\pi) = f(x) \ \ \forall x. \tag{11}$$

Let $\underline{f}$ be the discrete representation of the function $f(x)$ on an equidistant grid where $x_j = j\Delta x,\ \Delta x = 2\pi/N,\ j = 0, 1, \ldots, N-1$ with $N = 20$,

$$\underline{f} = [f_0, f_1, \ldots, f_{N-1}]^T \quad \text{where } f_j = f(x_j)\ . \tag{12}$$

a) Write a MATLAB script that computes the matrix $\underline{\underline{D}}$ corresponding to left-sided finite differences of first order. The matrix $\underline{\underline{D}}$ is defined as

$$\underline{f}'_{num} = \underline{\underline{D}}\ \underline{f} \tag{13}$$

with the vector $\underline{f}'_{num}$

$$\underline{f}'_{num} = [\delta f_0, \delta f_1, \ldots, \delta f_{N-1}]^T, \tag{14}$$

and the finite-difference operator δf_j

$$\delta f_j = \frac{f_j - f_{j-1}}{\Delta x}\ . \tag{15}$$

Remember that $f(x)$ is periodic when computing the derivative at the point $x = x_0$, i.e. $f_{-1} = f_{N-1}$.

b) Assume $f(x) = e^{ikx}$ and derive the expression for the modified wavenumber $\tilde{k}$ for the left-sided finite-difference scheme. Non-dimensionalise the wavenumber with the grid spacing, i.e. calculate $\Delta x\tilde{k}(k\Delta x)$.

c) From now on assume that $k = 5$ (i.e. a specific wave). Compute the derivative in a discrete (δf_j) and analytical ($f'_{x=x_j}$) manner in every grid point. Use the previously defined $\underline{\underline{D}}$ for the discrete derivative. Plot the real part for both the numerical and analytical derivative as a function of x.

d) Plot the real and imaginary part of the complex quantity

$$\mu_j = \frac{\delta f_j}{f_j} \tag{16}$$

as a function of x.

e) The result of part d) indicates that the vector corresponding to the discrete Fourier harmonic $\underline{F}_k = [e^{ikx_0} \quad e^{ikx_1} \quad \ldots \quad e^{ikx_{N-1}}]^T$ is in special relation with the matrix $\underline{\underline{D}}$. Can you say what that relation between $\underline{F}_k$ and $\underline{\underline{D}}$ is in terms of frequently used operators in linear algebra?

f) Compare the values of μ_j with the complex number $i\tilde{k}$ for the left-sided finite differences as derived in b). What can you say about the significance of μ_j?