Homework 5

Projection Method and Staggered Grid for Incompressible Flows

due 24/2-2014

The motivation for this exercise is to demonstrate the advantages of a staggered grid compared to a co-located one for incompressible flows.

When solving the incompressible Navier–Stokes equations, other techniques are normally used compared to those used for the solution of the compressible Navier–Stokes equations. One common method, used to ensure that the velocity field is divergence free, is the projection (or pressure correction) method. A detailed description of this method will be given later in the course. However, we use a simple version of it for our model equation.

A strongly simplified one-dimensional analogue to the incompressible Navier–Stokes equations neglecting the viscous and non-linear convection terms is given by

$$u_t = -p_x \, , \qquad (1)$$

$$u_x = 0 \, . \qquad (2)$$

Let us start with the discretisation of the time derivative using a first-order accurate backward Euler scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -(p_i^{n+1})_x \quad \Rightarrow \quad u_i^{n+1} = u_i^n - \Delta t \; (p_i^{n+1})_x \, . \qquad (3)$$

Taking the x-derivative of the equation above yields

$$(u_i^{n+1})_x = (u_i^n)_x - \Delta t \; (p_i^{n+1})_{xx} \, . \qquad (4)$$

Requiring $u_x = 0$ at $t = t^{n+1}$ results to

$$(p_i^{n+1})_{xx} = \frac{1}{\Delta t} \; (u_i^n)_x \, . \qquad (5)$$

Your task is

1. to write the discretised form of the equation (5), using central differences second order in space, for
   (a) a co-located grid,
   (b) a staggered grid (see figure 1);
2. to write a small program that solves the discretised equations derived above (for both grids).

Solve the equations with initial and boundary conditions

$$u(t = 0, x) = 1, \qquad p(t, x = 0) = 1 \, .$$

The spatial domain is $x \in [0, 1]$. **At each time step, p^{n+1} is computed first from equation (5), and then u^{n+1} is obtained from the equation (3)** (using central differences second order in x).

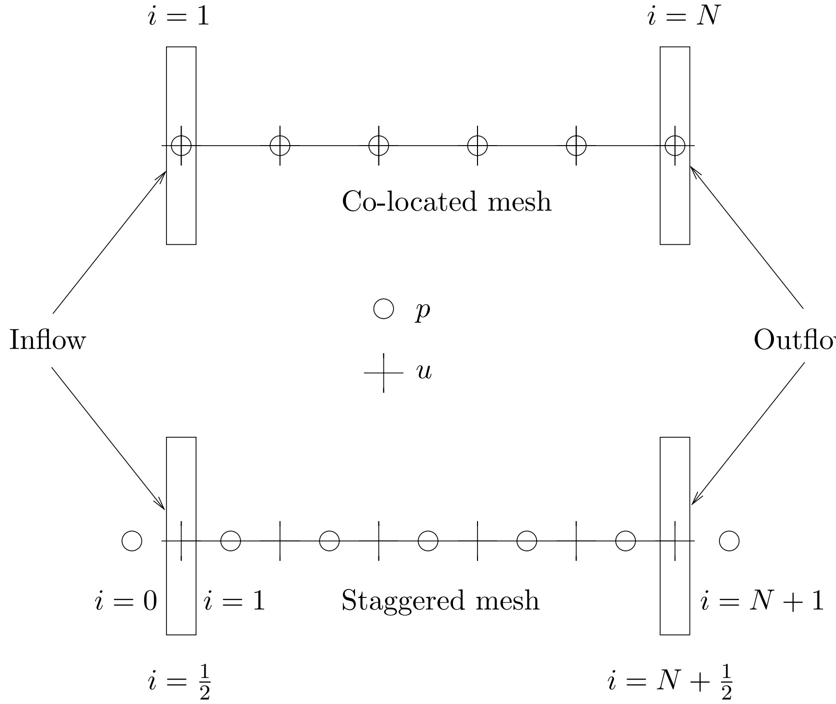

Figure 1: Schematics of a co-located and a staggered grid, respectively. Note the numbering of the nodes in both cases.

The implementation of the boundary conditions should be done very carefully. Note that for the given problem explicit boundary conditions for the velocities are not required. At the inflow Dirichlet boundary conditions for the pressure are given while at the outflow Neumann conditions should be used (first order, *i.e.* $p_{N-1} = p_N$ for the co-located grid and $p_{N+1} = p_N$ for the staggered one). Think how these two conditions should be applied. The values of p_0 and p_{N+1} can be approximated by linear extrapolation if necessary.

For the report, you need to

- For both co-located and staggered grid, write down explicitly all equations at the boundaries in the discretised form.
- Plot pressure and velocity field at the different time steps. Compare the results from (a) and (b) and explain the different behaviour of the solutions.

  The idea is to demonstrate the existence of spurious checker-board solutions. Something to keep in mind is that if the chosen initial solution is exact it might happen that you will not observe these spurious solutions. Therefore, perturb your initial condition with some low-amplitude noise ($\sim \mathcal{O}(0.01)$) in the *internal points* of the domain. The MATLAB function `rand` produces random numbers that can be used for that purpose.
- Use different $\Delta t/\Delta x$. What effects do you observe? Integrate the equations for long enough such that you can see whether the spurious solutions grow or disappear completely.