System Planning 2014

Lecture 7, F7: **Optimization**

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Course goals

Appendix A

Content:

- Generally about optimization
- Formulate optimization problems
- Linear Programming (LP)
- Mixed Integer Linear Programming (MILP)

Course Goals – Short term planning

- To pass the course,
  - the students should show that they are able to
  - formulate short-term planning problems of hydro-thermal power systems,
- To receive a higher grade
  - the students should also show that they are able to
  - create specialised models for short-term planning problems,
- Computation
  - Ability solving short-term planning problems essential
  - Computation, not a goal, not examined
  - Home assignments
    - give bonus points
    - some are mandatory

Optimization – Generally (1/2)

- Applied mathematics
- Application:
  - technology, science, economics, etc.
  - Something to be maximized or minimized
    - Profit, cost, energy, speed, losses
  - Given some constraints
  - Physical, technical, economical, legal, etc.

Optimization – Generally (2/2)

In general form

$$\begin{aligned} \min_{x} \quad & f(x), \\ \text{subject to} \quad & \left.\begin{aligned} g(x) &\le b \\ h(x) &= c \\ \underline{x} \le x &\le \overline{x} \end{aligned}\right\} \text{Constraints} \end{aligned}$$

- $f(x)$ **objective function**
- x **optimization variable** (multidimensional, e.g. vector)
- $g(x),\ h(x)$ **constraints** (multidimensional, e.g. vector)
- $\underline{x},\ \overline{x}$ **variable bounds** (like the variable)

Optimization – Formulating problems (1/6)

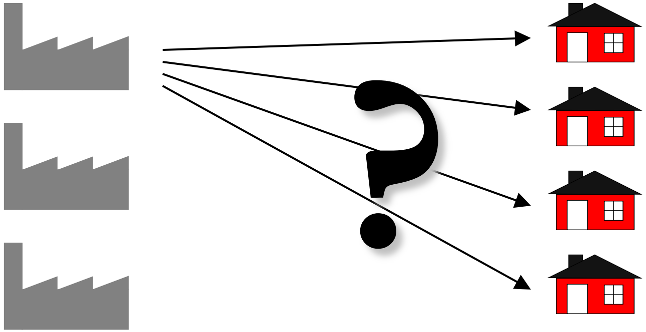
1. Formulate it verbally:
    1. Think the problem through
2. Define denotations
    1. Which parameters and variables are needed?
3. Formulate the problem in equation form
    1. That is; define the objective and constraints

(2 and 3 are interrelated)

Optimization – Formulating problems (2/6)

Example:

- A corporation owns a number of factories
- How to deliver items to consumers?



Optimization – Formulating problems (3/6)

1. Formulate verbally:
    1. How to deliver the goods, while
    2. Minimizing the transport costs?
    3. Subject to:
        1. Each factory's production capacity limit
        2. Fulfil the consumer demand

Optimization – Formulating problems (4/6)

2. Define denotations:
    1. Indices and parameters:
        1. *m* factories, factory *i* has the capacity a_i
        2. *n* customers, customer *j* demands b_j units
        3. Transportation costs from factory *i* to customer *j* is c_{ij} per unit
    2. Variables
        1. Let x_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ denote the number of units transported from factory *i* to customer *j*

Optimization – Formulating problems (5/6)

3. Formulate the problem mathematically:

Objective function

- Transportation cost: $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Constraints

- Production capacity: $\sum_{j=1}^{n} x_{ij} \leq a_i \qquad i = 1 \ldots m$
- Demand: $\sum_{i=1}^{m} x_{ij} \geq b_j \qquad j = 1 \ldots n$
- Variable bounds: $x_{ij} \geq 0 \qquad i = 1 \ldots m,\ j = 1 \ldots n$

Optimization – Formulating problems (6/6)

The optimization problem:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \leq a_i, \qquad i = 1 \ldots m$$

$$-\sum_{i=1}^{m} x_{ij} \leq -b_j, \qquad j = 1 \ldots n$$

$$x_{ij} \geq 0, \qquad i = 1 \ldots m,\ j = 1 \ldots n$$

LP – Generally (1/2)

- **Linear problems** (LP-problems)
  - A class of optimization problems
  - Linear objective, linear constraints
- All LPs may be formulated (standard form):

$$\begin{aligned} & \min c^T x \\ \text{subject to}\quad & Ax = b \\ & x \geq 0 \end{aligned}$$

LP – Generally (2/2)

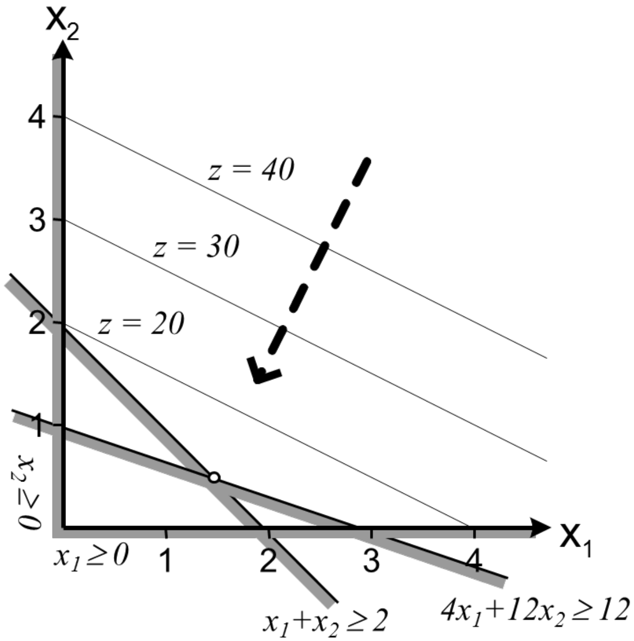
- Various types of LP-problems explained
- Examples similar to those in Appendix A
- We will study:
  - Extreme points
  - Slack variables
  - No feasible solution
  - Inactive constraints
  - Unbounded problem
  - Degenerate problem
  - Flat optimum
  - Dual formulation
  - Solution methods

LP – Basic Example (1/2)

The problem:

$$\begin{aligned}
&\text{min} && z = 5x_1 + 10x_2 \\
&\text{subject to} && x_1 + x_2 \geq 2 \\
& && 4x_1 + 12x_2 \geq 12 \\
& && x_1 \geq 0,\; x_2 \geq 0
\end{aligned}$$

LP – Basic Example (2/2)



Optimum at:

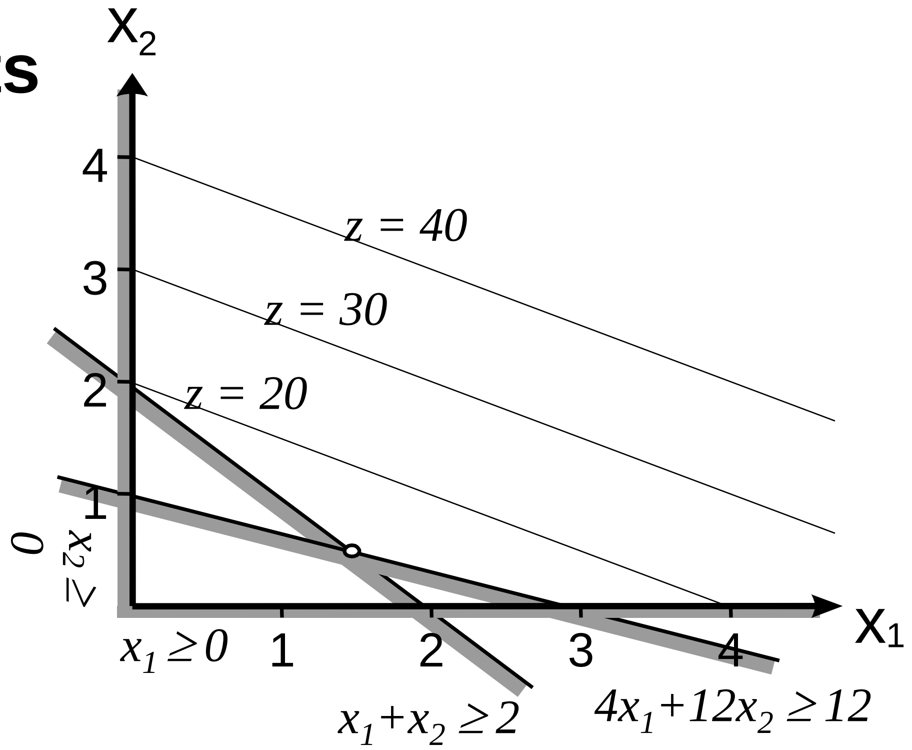
$$x_1 = 1.5$$
$$x_2 = 0.5$$

Objective value:

$$z = 12.5$$

LP – Extreme points

- “Corners” in feasible space
- Denoted *extreme points*
- Optimum always in extreme point(s)



LP – Slack variables (1/2)

- LP-problem in standard form:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Introduce *slack variables*
- All constraints can be equalities

LP – Slack variables (2/2)

- Without slack variables ($Ax \geq b$) :

$$
\begin{aligned}
&x_1 + x_2 \geq 2 \\
&4x_1 + 12x_2 \geq 12 \\
&x_1 \geq 0,\; x_2 \geq 0
\end{aligned}
$$

- With slack variables ($Ax = b$):

$$
\begin{aligned}
&x_1 + x_2 - x_3 = 2 \\
&4x_1 + 12x_2 - x_4 = 12 \\
&x_1 \geq 0,\; x_2 \geq 0,\; x_3 \geq 0,\; x_4 \geq 0
\end{aligned}
$$

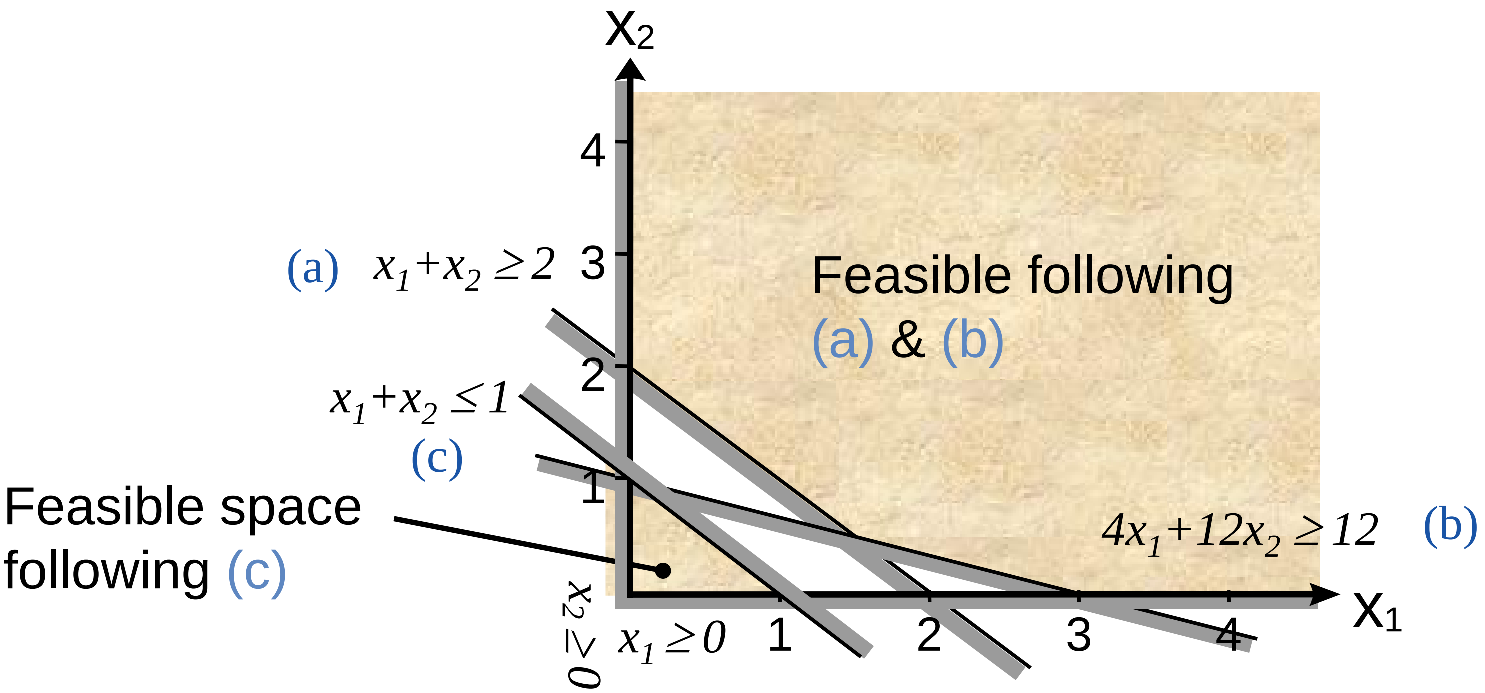
LP – Infeasible problem (1/2)

- Feasible space empty
- Bad formulation

$$
\begin{aligned}
\min \quad & z = 5x_1 + 10x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 2 \\
& 4x_1 + 12x_2 \geq 12 \\
& x_1 + x_2 \leq 1 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

Added constraint

LP – Infeasible problem (2/2)



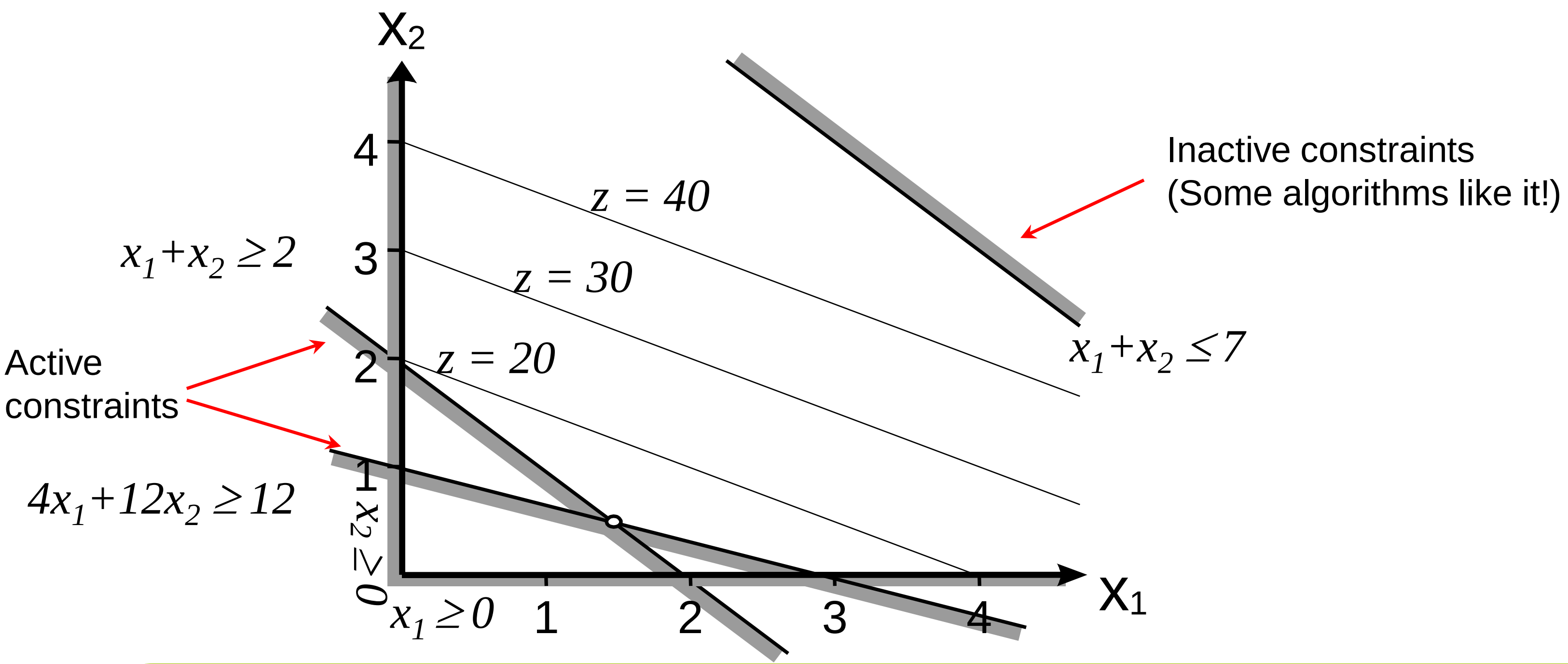
LP – Inactive Constraints (1/2)

- Some constraints does not constrain
- Given the objective

$$
\begin{aligned}
\min \quad & z = 5x_1 + 10x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 2 \\
& 4x_1 + 12x_2 \geq 12 \\
& x_1 + x_2 \leq 7 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

Added constraint

LP – Inactive Constraints (2/2)



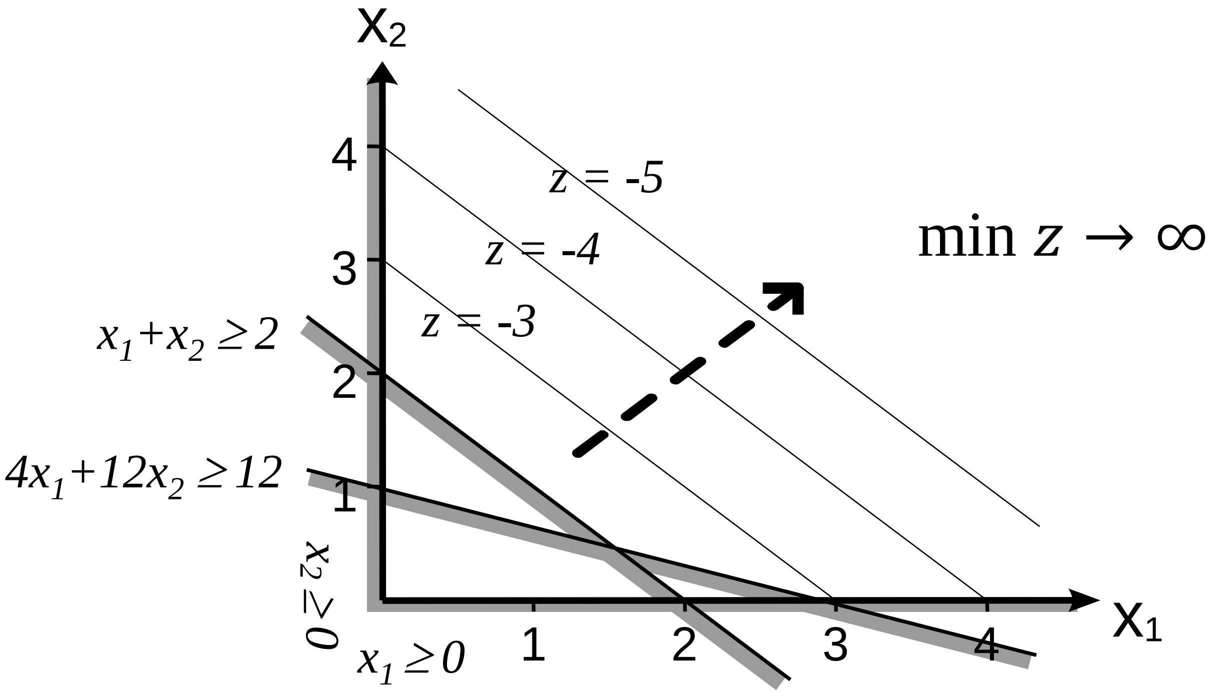
LP – Unbounded Problem (1/2)

- If no constraint is active
- The objective is unbounded
- $z \to \infty$

Objective function modified

$$
\begin{aligned}
\min \quad & z = -x_1 - x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 2 \\
& 4x_1 + 12x_2 \geq 12 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

LP – Unbounded Problem (2/2)



LP – Degenerate solution (1/2)

- More than one extreme point optimal
- All linear combination of these optimal

New objective function

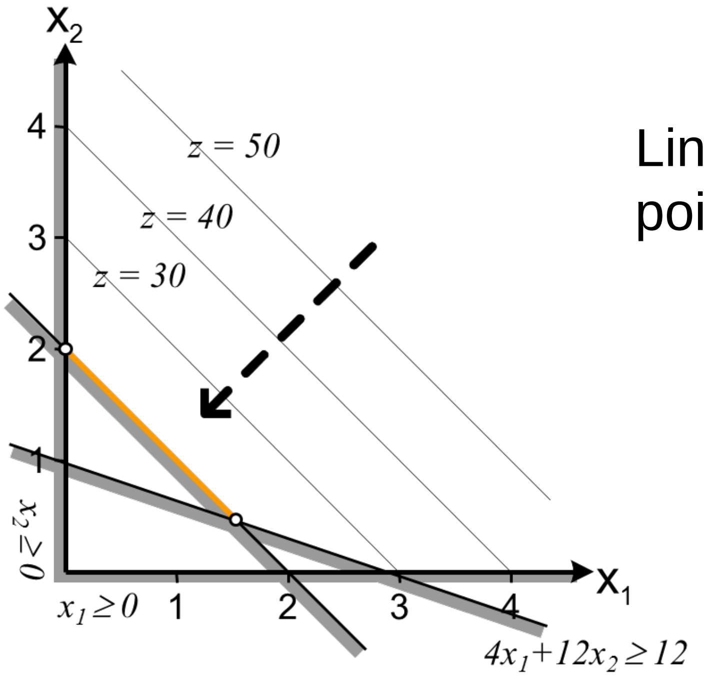
$$\min \quad z = 10x_1 + 10x_2$$

$$\text{subject to} \quad x_1 + x_2 \geq 2$$

$$4x_1 + 12x_2 \geq 12$$

$$x_1 \geq 0,\ x_2 \geq 0$$

LP – Degenerate solution (2/2)



Line between extreme points optimal

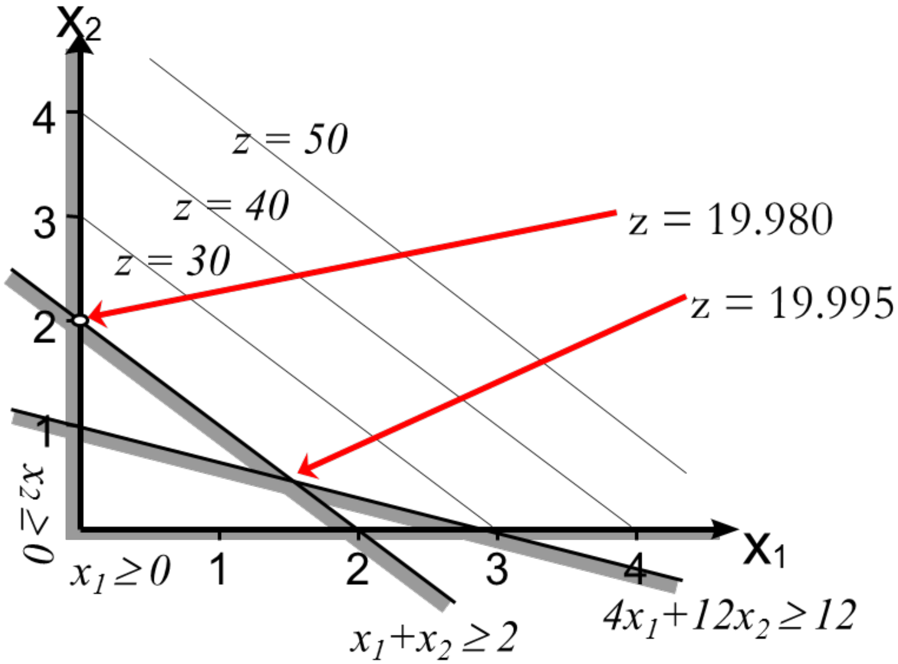
LP – Flat optimum (1/2)

Extreme points with similar objective values

$$
\begin{aligned}
\min \quad & z = 10x_1 + 9.99x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 2 \\
& 4x_1 + 12x_2 \geq 12 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

New objective function

LP – Flat optimum (2/2)



LP – Duality (1/5)

Any LP problem (*primal* problem) has a corresponding *dual* problem

Primal problem:		**Dual problem:**	
$\min_x$	$c^T x$	$\max_\lambda$	$b^T \lambda$
subject to	$Ax = b$	subject to	$A^T \lambda \le c$
	$x \ge 0$		$\lambda \ge 0$

λ is said to be a *dual variable*

LP – Duality (2/5)

Theorem (strong duality):

”If the primal problem has an optimal solution, then also the dual problem has an optimal solution. The objective values of these two problems equals.”

LP – Duality (3/5)

$$\min_x \quad c^T x$$

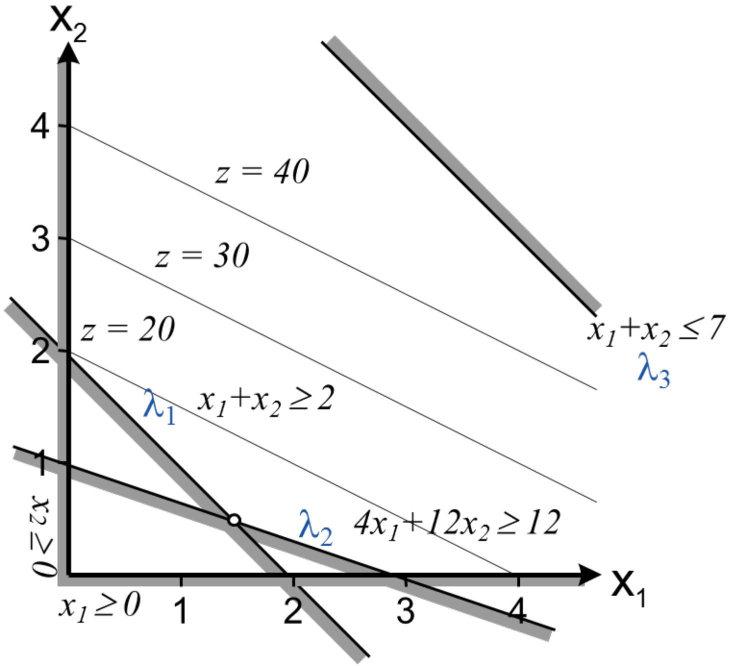
$$\text{subject to} \quad Ax = b$$

$$x \geq 0$$

One dual variable for each constraint!

- Question:
  - What dual variables technically describe?
- Answer:
  - Marginal value of the corresponding constraint
  - i.e. objective value's dependence on RHS'

LP – Duality (4/5)



In optimum:

$\lambda_1 > 0$ (active)

$\lambda_2 > 0$ (active)

$\lambda_3 = 0$ (inactive)

LP – Duality (5/5)

- Dual variables: λ
- **Small** perturbations in the right-hand-side, Δb
- $\Rightarrow$
- Changes in the objective value, Δz:

$$\Delta z = \lambda^T \Delta b$$

MILP – Generally

- Mixed Integer Linear Programming problems
- Class of optimization problems
- Linear objective functions and linear constraints
- Some variables may be integers

$$\min_x z = c^T x$$

$$\text{such that} \quad Ax = b$$

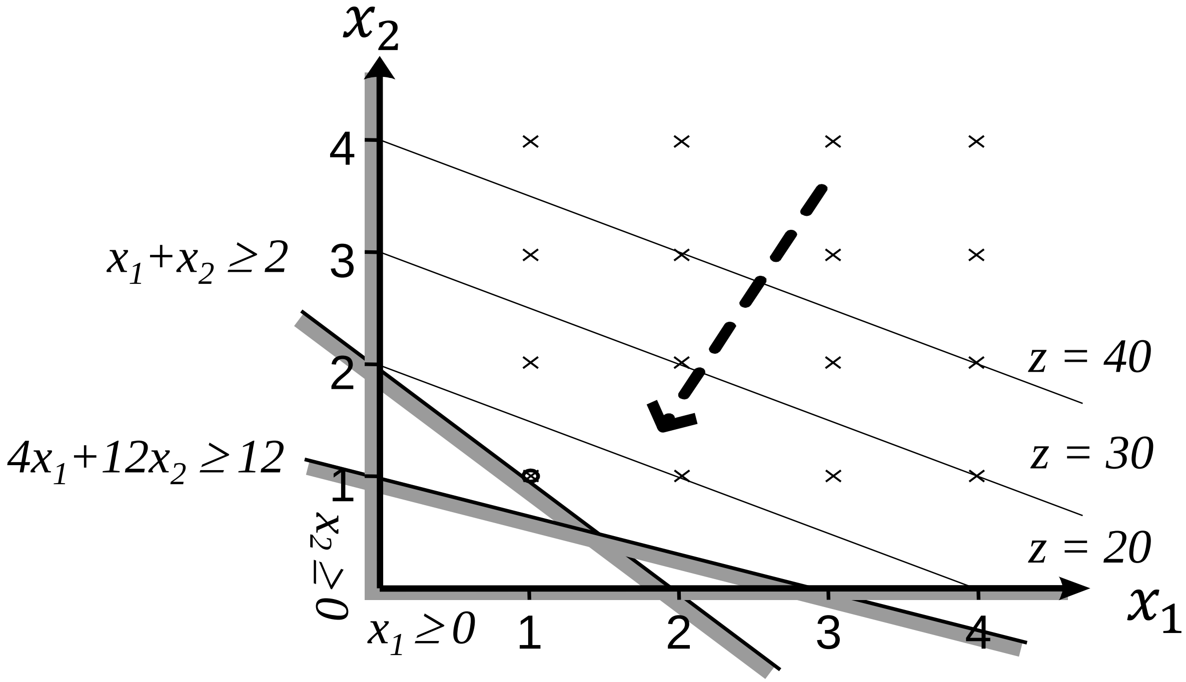
$$x \geq 0$$

$$x_i \in \mathbb{R}, i \in \{1, 2, \ldots, m\}$$

$$x_i \in \mathbb{Z}, i \in \{m+1, m+2, \ldots, n\}$$

$$|i| = n$$

MILP – Example



Optimum at:

$x_1 = 1 \quad (1.5)$

$x_2 = 1 \quad (0.5)$

Optimal objective:

$z = 15 \quad (12.5)$

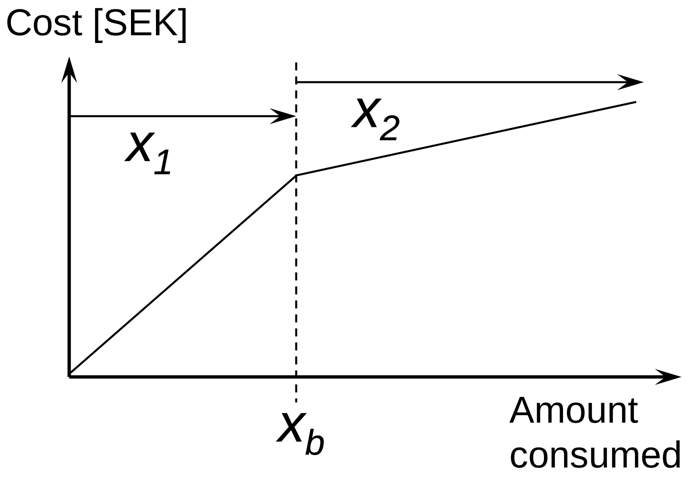
MILP – Solution

- Integer variables are easily implemented
- Integer problems generally hard to solve
- Computation time may increase exponentially
- If possible, avoid integer variables!
- Special case integer variable: binary variable

$$x \in \{0,1\}$$

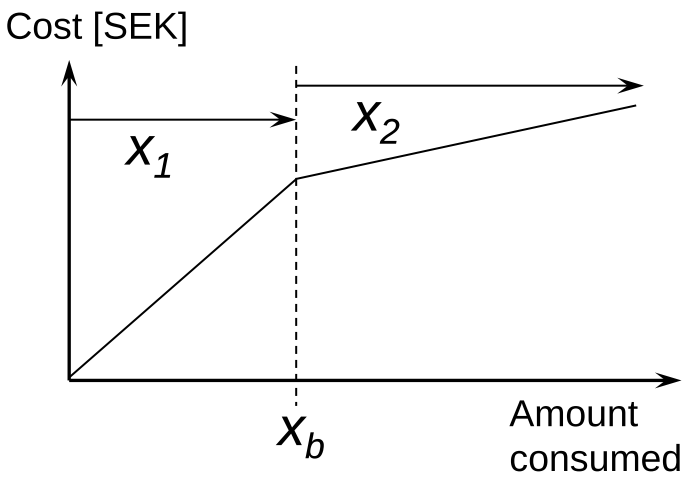
MILP – Applying Binary Variable (1/6)

- Minimize cost buying certain product
- Variable, $x \geq 0$: amount of the product,
- Decreasing marginal cost, e.g.:
  - Economies of scale
  - (outside this course, square root)
  - Volume discount



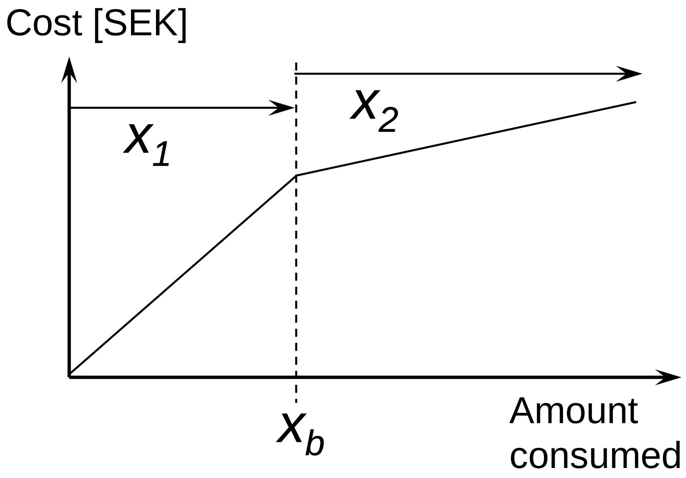
MILP – Applying Binary Variable (2/6)

- Split x up in two variables
- $x = x_1 + x_2$
- $x_1 \leq x_b$
- $z = c_1 x_1 + c_2 x_2$
- $c_1 > c_2$



MILP – Applying Binary Variable (3/6)

- Segment 2 cheaper
- Prevent x_2 being positive as long
- $x_1 < x_b$
- Typically done by using binary variable



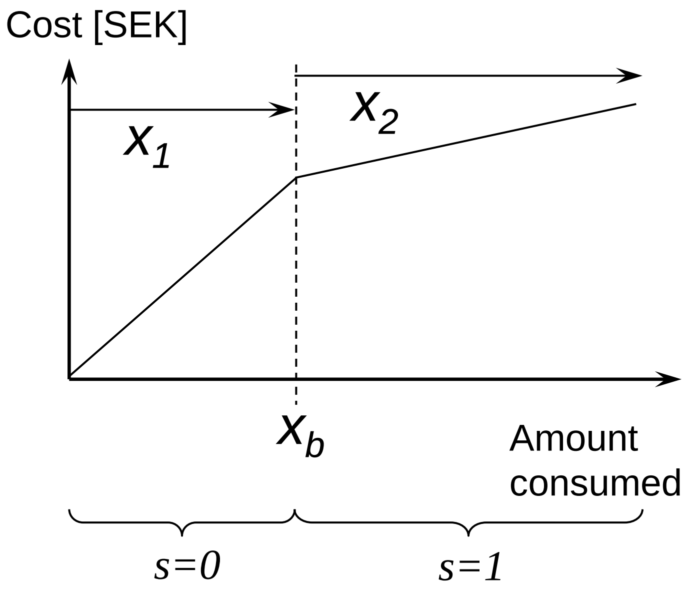
MILP – Applying Binary Variable (4/6)

- Introduce the binary variable s
- And the constraints

$$x_1 - x_b s \geq 0$$

$$-x_2 + Ms \geq 0$$

- Note: In real-life, M not arbitrary



MILP – Applying Binary Variable (5/6)

- Sizing of parameter M
- Not a part of the course

$$\mathrm{M} = \min_{m} m,$$

subject to:

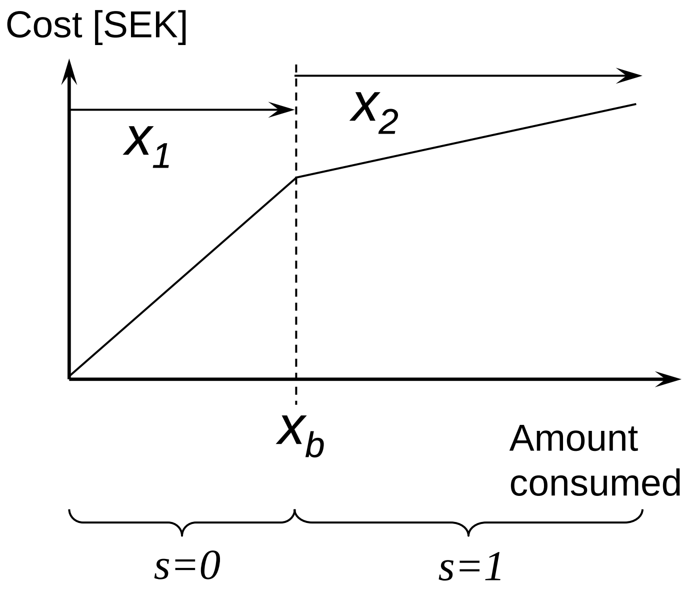
$$\min_{x_2} z = \left(-x_2 + m\right)$$

$$\left(-x_2 + m\right) \geq 0$$

$$\mathrm{m} > 0$$

- Or, simpler

$$\mathrm{M} = -\min_{x_2} \left(-x_2\right)$$



MILP – Applying Binary Variable (6/6)

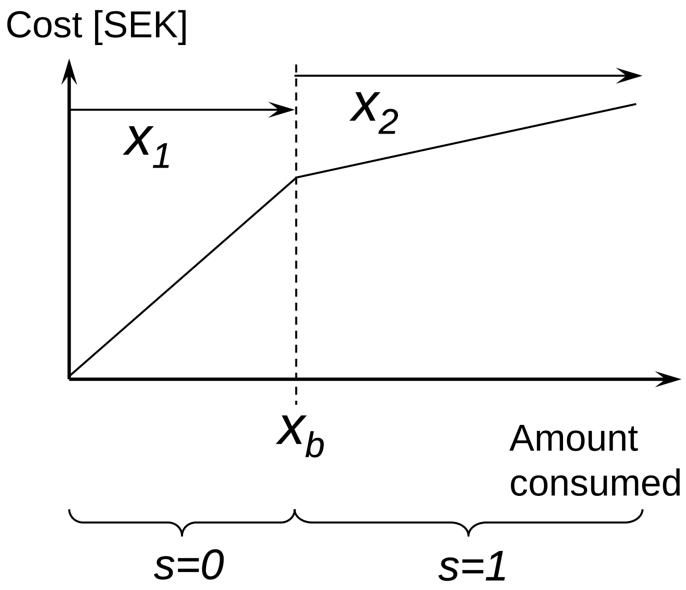
$$x_1 - x_b s \geq 0$$

$$-x_2 + Ms \geq 0$$

If $s=0$: $x_1 \geq 0, \quad -x_2 \geq 0 \Leftrightarrow x_2 \leq 0 \Rightarrow x_2 = 0$

If $s=1$: $x_1 - x_b \geq 0, \Leftrightarrow x_1 \geq x_b \Rightarrow x_1 = x_b$

$-x_2 + M \geq 0 \Leftrightarrow x_2 \leq M$



End of lecture 7

Next time short-term planning