

VEKTORANALYS

Kursvecka 5

CURVILINEAR COORDINATES

Kapitel 10

Sidor 99-121

TARGET PROBLEM

- An athlete is rotating a hammer
- Calculate the force on the arms.

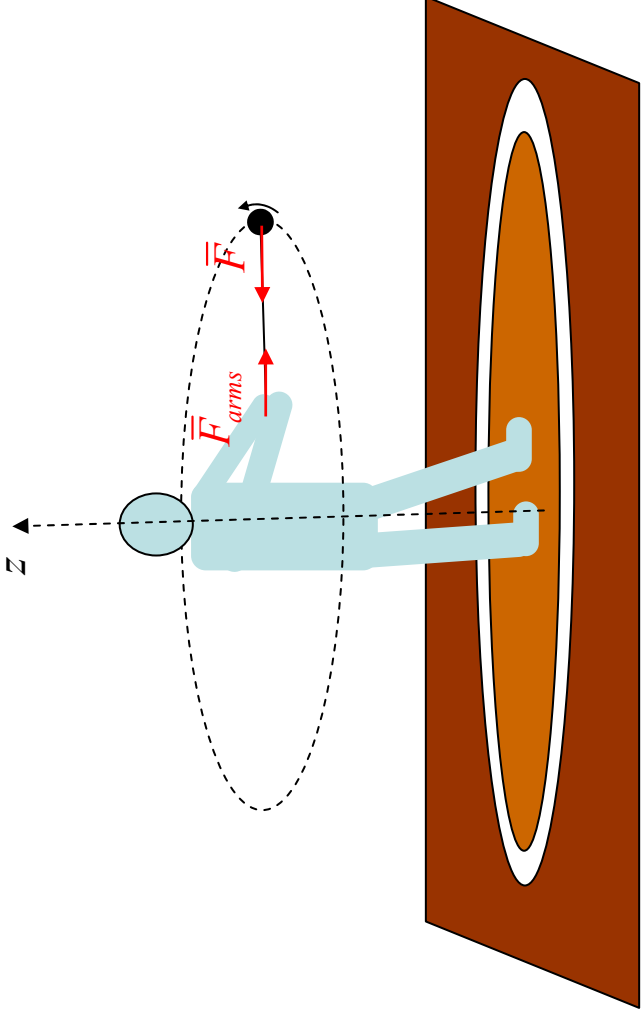
$$\bar{\mathbf{F}}_{arms} = -\bar{\mathbf{F}}$$

$$\bar{\mathbf{F}} = m\bar{\mathbf{a}}$$

$$\bar{\mathbf{a}} = \frac{d\bar{\mathbf{v}}}{dt} \equiv \dot{\bar{\mathbf{v}}}$$

$$\bar{\mathbf{v}} = \frac{d\bar{\mathbf{r}}}{dt} \equiv \dot{\bar{\mathbf{r}}}$$

$$\bar{\mathbf{r}} = \rho_0 \hat{\mathbf{e}}_\rho + z_0 \hat{\mathbf{e}}_z \quad (\text{in cylindrical coord.})$$



We need

- to introduce **curvilinear coordinates**
- to describe **cylindrical coordinates**
- to calculate **the derivative of $\hat{\mathbf{e}}_\rho$**

CYLINDRICAL COORDINATES

Cylindrical coordinates are an example of curvilinear coordinates

cartesian coord.

P : x_0, y_0, z_0

cylindrical coord.

P : ρ, φ, z_0

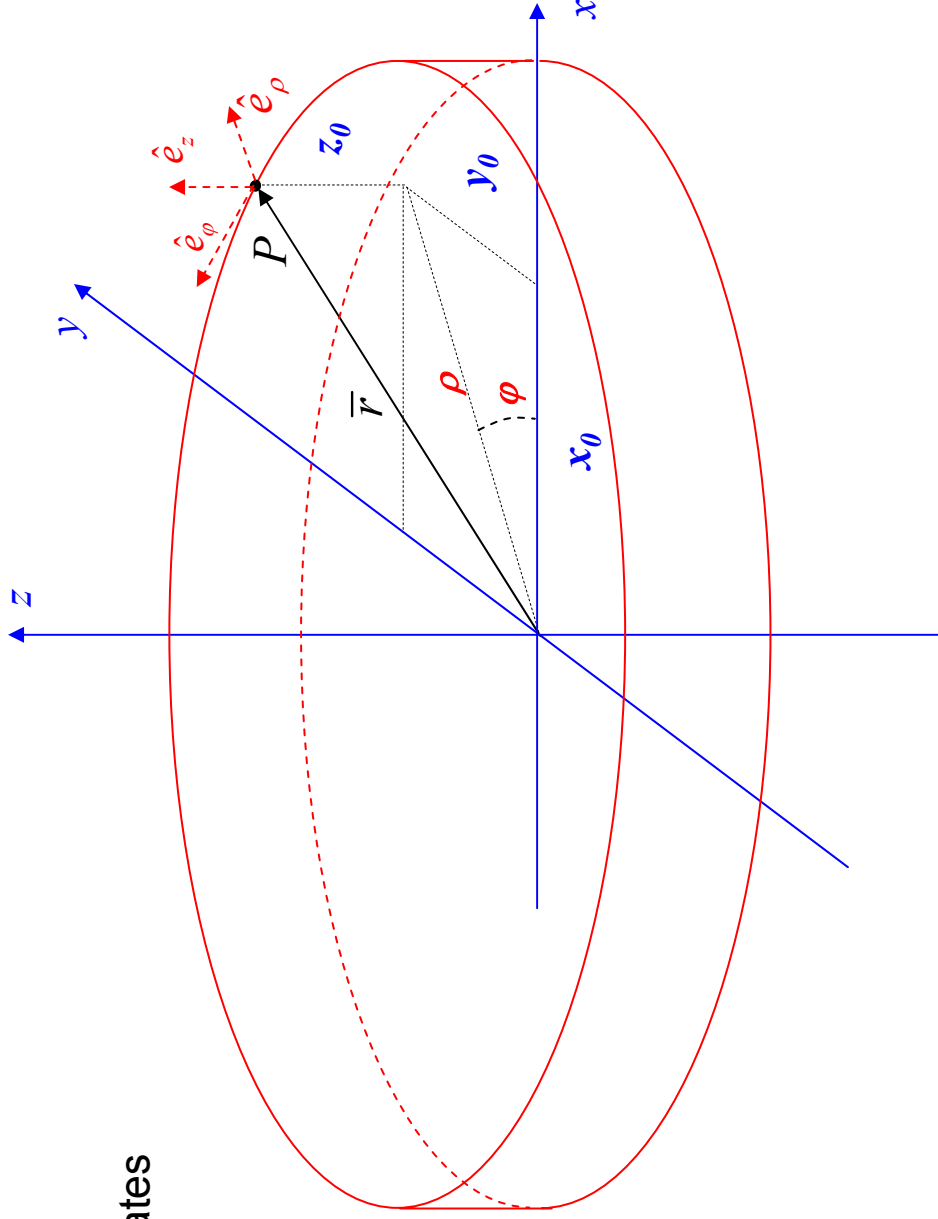
$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \tan \varphi = y / x \\ z = z \end{cases}$$

$$0 \leq \rho \leq \infty$$

$$0 \leq \varphi \leq 2\pi$$

$$-\infty \leq z \leq +\infty$$

$$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \\ z = z \end{cases}$$



CURVILINEAR COORDINATES

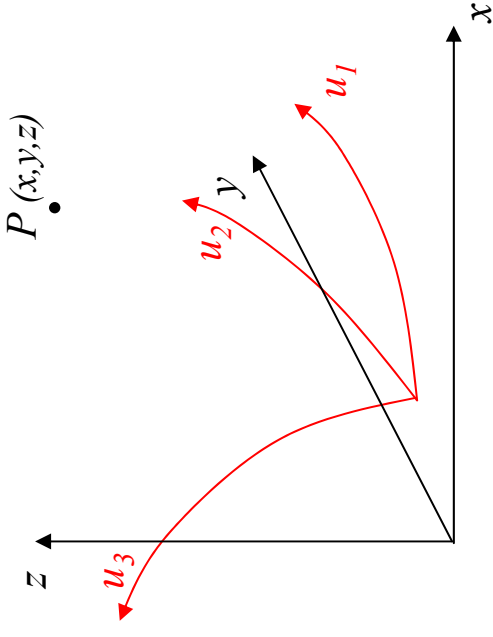
Consider a cartesian coordinate system (x, y, z) and then another coordinate system (u_1, u_2, u_3) , which will be called **curvilinear**

- The 3 curves u_1, u_2, u_3 are the coordinate axes
- The surfaces defined by $u_i = \text{const.}$ are called **coordinate surfaces**
- The curves defined by the intersection of the coordinate surfaces are called **coordinate curves**

A point P has coordinates (x, y, z) in the cartesian and coordinates (u_1, u_2, u_3) in the curvilinear

We assume that there is a one-to-one relationship between x_i and u_i , so that x_i can be expressed as a function of u_i (and vice-versa):

$$\begin{cases} x = x(u_1, u_2, u_3) \\ y = y(u_1, u_2, u_3) \\ z = z(u_1, u_2, u_3) \end{cases}$$



DEFINITION

A system of curvilinear coordinates is orthogonal if the coordinate curves are perpendicular to each other where they intersect

$$\bar{u}_i \cdot \bar{u}_j = \delta_{ik}$$

CURVILINEAR COORDINATES

Point P

the basis is defined by
the unit vectors:

$$d\vec{r}$$

x, y, z

$$\hat{e}_x \quad \hat{e}_y \quad \hat{e}_z$$

$$(dx, dy, dz)$$

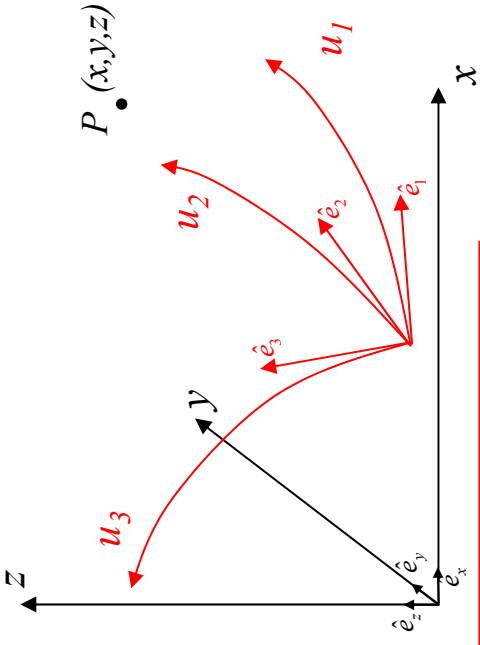
u_1, u_2, u_3

?

?

CARTESIAN

CURVILINEAR



An orthogonal curvilinear coordinate system has an orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ in each point and

DEFINITION

$$\hat{e}_i = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial u_i} \quad \text{with scale factor } h_i = \left| \frac{\partial \vec{r}}{\partial u_i} \right|$$

$$\text{Orthonormal: } \left\{ \begin{array}{l} \text{magnitude 1} \quad |\hat{e}_i| = \left| \frac{1}{h_i} \frac{\partial \vec{r}}{\partial u_i} \right| = \frac{1}{|h_i|} \left| \frac{\partial \vec{r}}{\partial u_i} \right| = 1 \\ \text{orthogonal} \quad \hat{e}_i \cdot \hat{e}_j = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial u_i} \cdot \frac{1}{h_j} \frac{\partial \vec{r}}{\partial u_j} = 0 \quad \text{for } i \neq j \end{array} \right. \Rightarrow \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

Kronecker delta

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 = h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2 du_2 + h_3 \hat{e}_3 du_3$$

$$d\vec{r} = \sum_i h_i du_i \hat{e}_i$$

SURFACE ELEMENT AND VOLUME ELEMENT

In a Cartesian coordinate system:

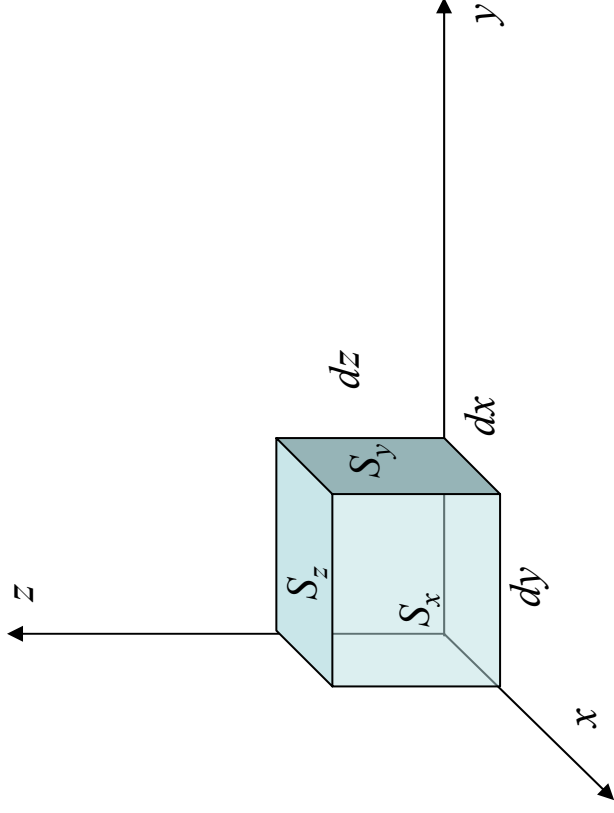
$$d\vec{r} = (dx, dy, dz) = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z$$

$$dS_x = dydz$$

$$dS_y = dxdz$$

$$dS_z = dxdy$$

$$dV = dxdydz$$



In an orthogonal curvilinear coordinate system:

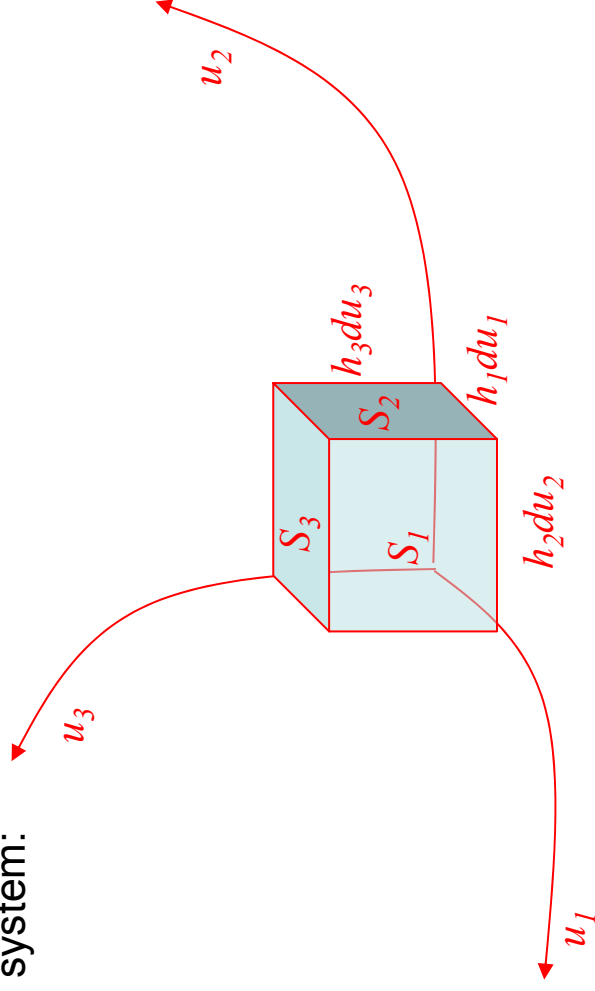
$$d\vec{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$$dS_1 = h_2 h_3 du_2 du_3$$

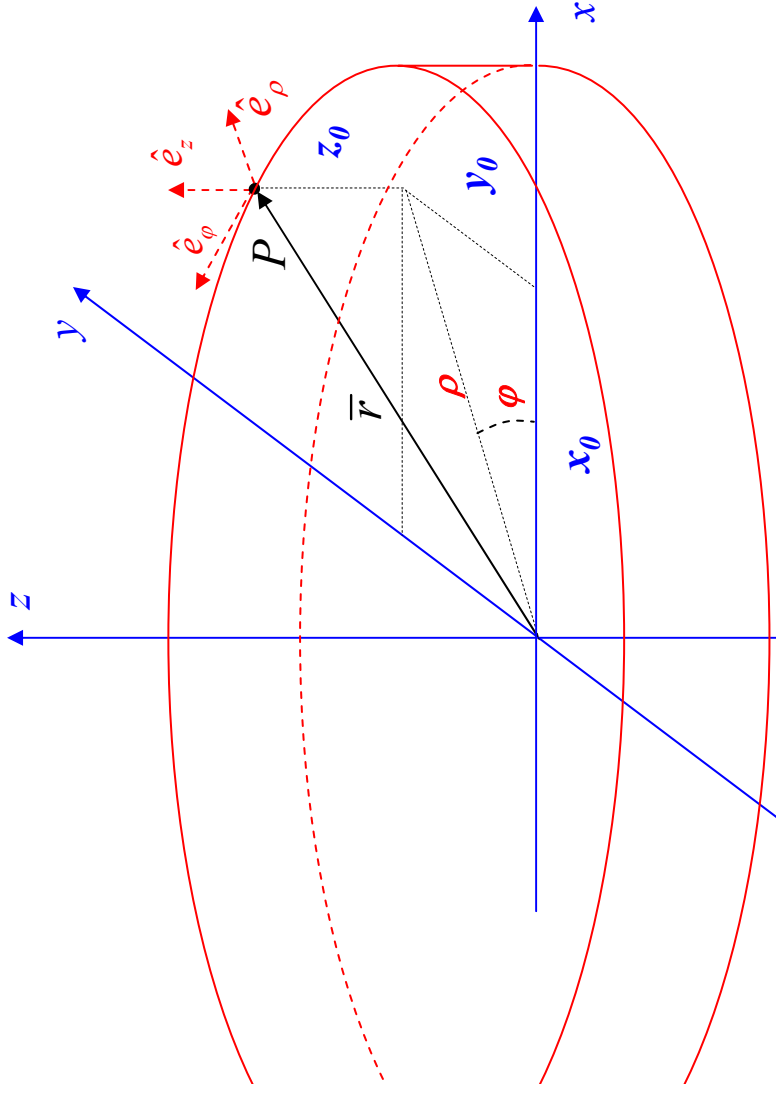
$$dS_2 = h_1 h_3 du_1 du_3$$

$$dS_3 = h_1 h_2 du_1 du_2$$

$$dV = h_1 h_2 h_3 du_1 du_2 du_3$$



CYLINDRICAL COORDINATES



Orthonormal basis

$$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \\ z = z \end{cases}$$

$$\vec{r} = \rho \cos \varphi \hat{e}_x + \rho \sin \varphi \hat{e}_y + z \hat{e}_z$$

$$\hat{e}_i = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial u_i} \quad \text{with} \quad h_i = \left| \frac{\partial \vec{r}}{\partial u_i} \right|$$

$$\left\{ \begin{array}{l} \frac{\partial \vec{r}}{\partial \rho} = (\cos \varphi, \sin \varphi, 0) \\ \frac{\partial \vec{r}}{\partial \varphi} = (-\rho \sin \varphi, \rho \cos \varphi, 0) \\ \frac{\partial \vec{r}}{\partial z} = (0, 0, 1) \end{array} \right.$$

$$\left\{ \begin{array}{l} h_\rho = \sqrt{\cos^2 \varphi + \sin^2 \varphi} = 1 \\ h_\varphi = \sqrt{(-\rho \sin \varphi)^2 + (\rho \cos \varphi)^2} = \rho \\ h_z = 1 \end{array} \right.$$

$$\left\{ \begin{array}{l} \hat{e}_\rho = \frac{1}{h_\rho} \frac{\partial \vec{r}}{\partial \rho} = (\cos \varphi, \sin \varphi, 0) = \cos \varphi \hat{e}_x + \sin \varphi \hat{e}_y \\ \hat{e}_\varphi = \frac{1}{h_\varphi} \frac{\partial \vec{r}}{\partial \varphi} = (-\sin \varphi, \cos \varphi, 0) = -\sin \varphi \hat{e}_x + \cos \varphi \hat{e}_y \\ \hat{e}_z = \frac{1}{h_z} \frac{\partial \vec{r}}{\partial z} = (0, 0, 1) = \hat{e}_z \end{array} \right.$$

EXERCISE: write \vec{r} using $\hat{e}_\rho, \hat{e}_\varphi, \hat{e}_z$

$$dS_\rho = \rho d\varphi dz$$

$$dS_z = \rho d\varphi d\rho$$

$$dV = \rho d\rho d\varphi dz$$

$$\begin{aligned}\bar{\mathbf{F}}_{\text{arms}} &= -\bar{\mathbf{F}} \\ \bar{\mathbf{F}} &= m\bar{\mathbf{a}}\end{aligned}$$

$$\bar{\mathbf{a}} = \frac{d\bar{\mathbf{v}}}{dt} \equiv \dot{\bar{\mathbf{v}}}$$

$$\bar{\mathbf{v}} = \frac{d\bar{\mathbf{r}}}{dt} \equiv \dot{\bar{\mathbf{r}}}$$

$$\bar{\mathbf{r}} = \rho_0 \hat{\mathbf{e}}_\rho + z_0 \hat{\mathbf{e}}_z \quad (\text{in cylindrical coord.})$$

$$\bar{\mathbf{F}}_{\text{arms}} = -m\bar{\mathbf{a}} = -m\dot{\bar{\mathbf{v}}} = -m\ddot{\bar{\mathbf{r}}}$$

$$\ddot{\bar{\mathbf{r}}} = \frac{d}{dt} \dot{\bar{\mathbf{r}}} = \frac{d}{dt} \left(\overset{=0 \text{ (the length does not change)}}{\dot{\rho}_0 \hat{\mathbf{e}}_\rho} + \rho_0 \dot{\hat{\mathbf{e}}}_\rho + \overset{=0 \text{ (the hammer rotates on a plane } z=\text{constant)}}{\dot{z}_0 \hat{\mathbf{e}}_z} + z_0 \dot{\hat{\mathbf{e}}}_z \right) =$$

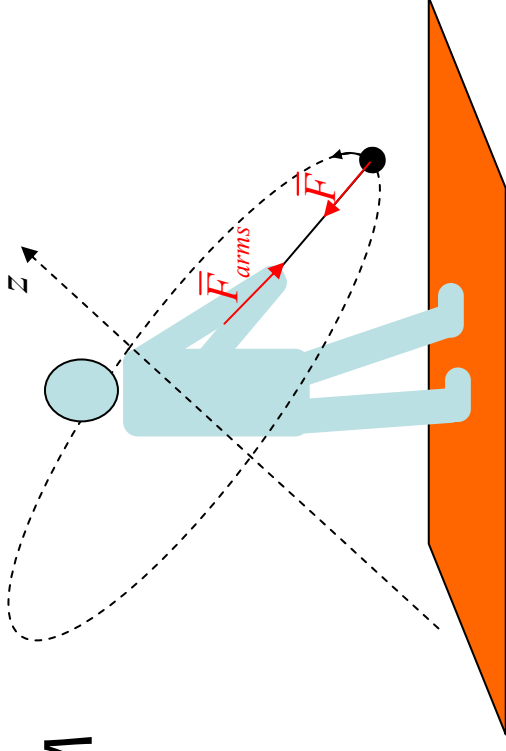
$$\dot{\hat{\mathbf{e}}}_\rho = \frac{d}{dt} (\cos \varphi, \sin \varphi, 0) = (-\dot{\varphi} \sin \varphi, \dot{\varphi} \cos \varphi, 0) = \dot{\varphi} (-\sin \varphi, \cos \varphi, 0) = \dot{\varphi} \hat{\mathbf{e}}_\varphi$$

$$\dot{\hat{\mathbf{e}}}_z = \frac{d}{dt} (0, 0, 1) = 0$$

$$= \frac{d}{dt} (\rho_0 \dot{\varphi} \hat{\mathbf{e}}_\varphi) = (\dot{\rho}_0 \dot{\varphi} \hat{\mathbf{e}}_\varphi + \rho_0 \ddot{\varphi} \hat{\mathbf{e}}_\varphi + \rho_0 \dot{\varphi} \dot{\hat{\mathbf{e}}}_\varphi) = (\dot{\rho}_0 \dot{\varphi} \hat{\mathbf{e}}_\varphi + \rho_0 \ddot{\varphi} \hat{\mathbf{e}}_\varphi - \rho_0 \dot{\varphi} \dot{\hat{\mathbf{e}}}_\varphi) = \rho_0 (\ddot{\varphi} \hat{\mathbf{e}}_\varphi - \dot{\varphi}^2 \hat{\mathbf{e}}_\rho)$$

$$\dot{\hat{\mathbf{e}}}_\varphi = \frac{d}{dt} (-\sin \varphi, \cos \varphi, 0) = (-\dot{\varphi} \cos \varphi, -\dot{\varphi} \sin \varphi, 0) = -\dot{\varphi} (\cos \varphi, \sin \varphi, 0) = -\dot{\varphi} \hat{\mathbf{e}}_\rho$$

$$\bar{\mathbf{F}}_{\text{arms}} = -m\ddot{\bar{\mathbf{r}}} = m\rho_0 (\dot{\varphi}^2 \hat{\mathbf{e}}_\rho - \ddot{\varphi} \hat{\mathbf{e}}_\varphi)$$



TARGET PROBLEM

WHICH STATEMENT IS WRONG?

- 1- \hat{e}_ρ does not depend on the position (yellow)
- 2- \hat{e}_z is constant everywhere (red)
- 3- Cylindrical coordinates are appropriate with a cylindrical symmetry (green)
- 4- Cylindrical coordinates are a curvilinear coordinate system (blue)

TARGET PROBLEM

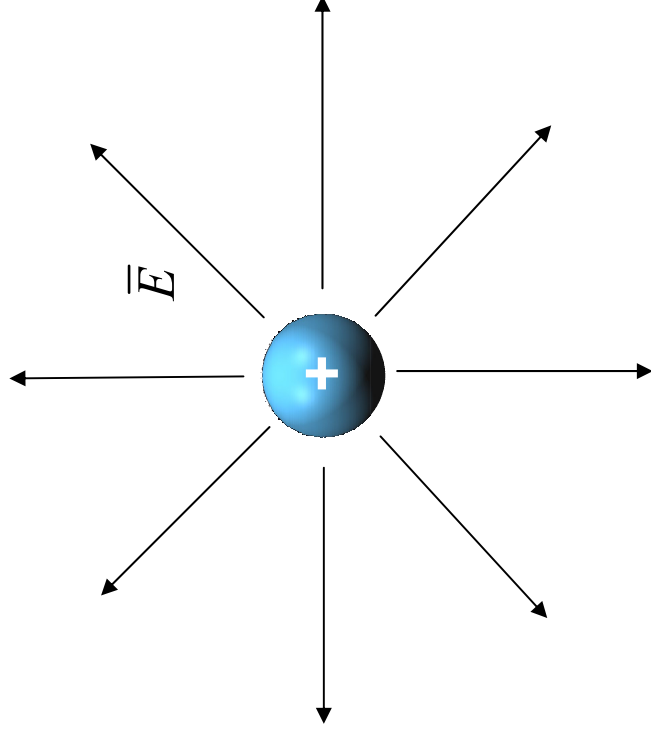
A vector field that is conservative has a potential: $\vec{E} = \nabla \phi$ (see theorems in week 2)

If \vec{E} is the electric field, $-\phi$ is called electrostatic potential.

$$\left. \begin{array}{l} \vec{E} = \nabla \phi \\ \nabla \cdot \vec{E} = 0 \end{array} \right\} \iff \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0$$

first Maxwell's equation with no charge

Laplace's equation



Calculate the electrostatic potential generated outside a spherical charge.

Due to the spherical symmetry, the solution will depend only on the radius: $\phi = \phi(r)$

with $r = |\vec{r}|$

We need to:

- introduce **spherical coordinates**
- calculate **gradient and divergence in spherical coordinates**
- solve the equation

SPHERICAL COORDINATES

spherical coordinates are an example of curvilinear coordinates

cartesian coord.

$P: x_0, y_0, z_0$

spherical coord.

$P: r, \theta, \varphi$

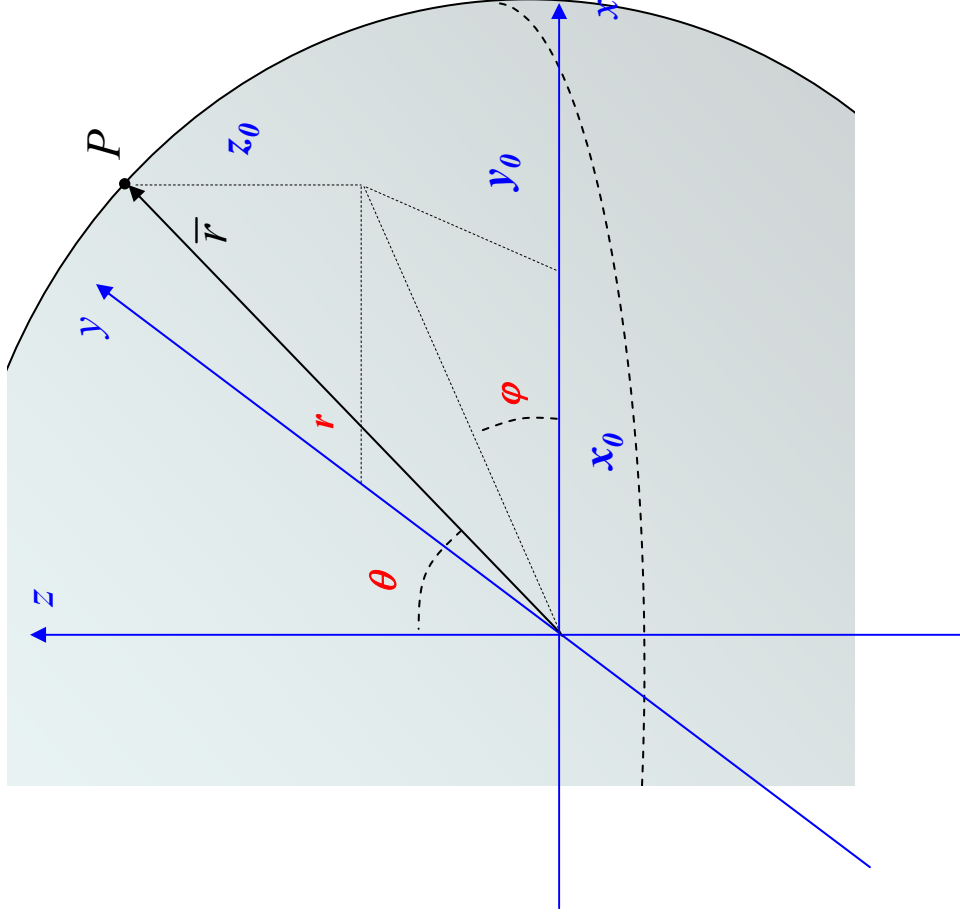
$$\left\{ \begin{array}{l} r = \sqrt{x^2 + y^2 + z^2} \\ \tan \theta = \frac{\sqrt{x^2 + y^2}}{z} \\ \tan \varphi = y/x \end{array} \right.$$

$$0 \leq r \leq \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \varphi \leq 2\pi$$

$$\left\{ \begin{array}{l} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{array} \right.$$



SPHERICAL COORDINATES

Orthonormal basis

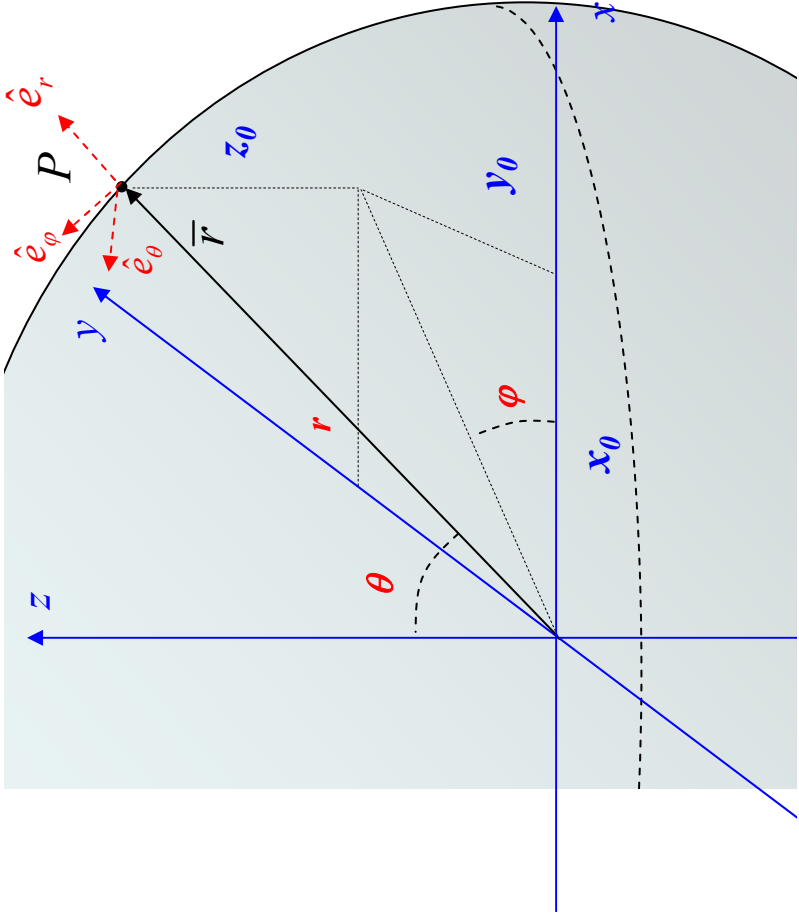
$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

$$\vec{r} = r \sin \theta \cos \varphi \hat{e}_x + r \sin \theta \sin \varphi \hat{e}_y + r \cos \theta \hat{e}_z$$

$$\hat{e}_i = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial u_i} \quad \text{with} \quad h_i = \left| \frac{\partial \vec{r}}{\partial u_i} \right|$$

$$\begin{cases} \frac{\partial \vec{r}}{\partial r} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \\ \frac{\partial \vec{r}}{\partial \theta} = (r \cos \theta \cos \varphi, r \cos \theta \sin \varphi, -r \sin \theta) \\ \frac{\partial \vec{r}}{\partial \varphi} = (-r \sin \theta \sin \varphi, r \sin \theta \cos \varphi, 0) \end{cases}$$

$$\begin{cases} h_r = \sqrt{\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta} = 1 \\ h_\theta = \sqrt{(r \cos \theta \cos \varphi)^2 + (r \cos \theta \sin \varphi)^2 + (r \sin \theta)^2} = r \\ h_\varphi = \sqrt{(r \sin \theta \sin \varphi)^2 + (r \sin \theta \cos \varphi)^2} = r \sin \theta \end{cases}$$



$$\begin{cases} \hat{e}_r = \frac{1}{h_r} \frac{\partial \vec{r}}{\partial r} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \\ \hat{e}_\theta = \frac{1}{h_\theta} \frac{\partial \vec{r}}{\partial \theta} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta) \\ \hat{e}_\varphi = \frac{1}{h_\varphi} \frac{\partial \vec{r}}{\partial \varphi} = (-\sin \varphi, \cos \varphi, 0) \end{cases}$$

$$\begin{aligned} dS_r &= r^2 \sin \theta d\theta d\varphi \\ dV &= r^2 \sin \theta dr d\theta d\varphi \end{aligned}$$

EXERCISE: write \vec{r} using $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$

GRADIENT IN CURVILINEAR COORDINATES

In cartesian coordinates:
$$\text{grad}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right) = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z$$

And in curvilinear coordinates?

We must express $\text{grad}\phi$ in terms of the curvilinear basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$:

$$\text{grad}\phi = g_1\hat{e}_1 + g_2\hat{e}_2 + g_3\hat{e}_3$$

since $d\phi = \text{grad}\phi \cdot d\vec{r}$ and $d\vec{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$

$$d\phi = (g_1\hat{e}_1 + g_2\hat{e}_2 + g_3\hat{e}_3) \cdot (h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3) = g_1 h_1 du_1 + g_2 h_2 du_2 + g_3 h_3 du_3$$

But also, writing ϕ as a function of u_i : $d\phi = \frac{\partial\phi}{\partial u_1} du_1 + \frac{\partial\phi}{\partial u_2} du_2 + \frac{\partial\phi}{\partial u_3} du_3$

Therefore: $g_1 = \frac{1}{h_1} \frac{\partial\phi}{\partial u_1}, \quad g_2 = \frac{1}{h_2} \frac{\partial\phi}{\partial u_2}, \quad g_3 = \frac{1}{h_3} \frac{\partial\phi}{\partial u_3}$

$$\text{grad}\phi = \sum_i \frac{1}{h_i} \frac{\partial\phi}{\partial u_i} \hat{e}_i$$

GRADIENT IN CURVILINEAR COORDINATES

THE GRADIENT

- in cylindrical coordinates:

$$\text{grad}\phi = \frac{1}{h_\rho} \frac{\partial\phi}{\partial\rho} \hat{e}_\rho + \frac{1}{h_\varphi} \frac{\partial\phi}{\partial\varphi} \hat{e}_\varphi + \frac{1}{h_z} \frac{\partial\phi}{\partial z} \hat{e}_z = \left(\frac{\partial\phi}{\partial\rho}, \frac{1}{\rho} \frac{\partial\phi}{\partial\varphi}, \frac{\partial\phi}{\partial z} \right)$$

- in spherical coordinates:

$$\text{grad}\phi = \frac{1}{h_r} \frac{\partial\phi}{\partial r} \hat{e}_r + \frac{1}{h_\theta} \frac{\partial\phi}{\partial\theta} \hat{e}_\theta + \frac{1}{h_\varphi} \frac{\partial\phi}{\partial\varphi} \hat{e}_\varphi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r} \frac{\partial\phi}{\partial\theta}, \frac{1}{r \sin\theta} \frac{\partial\phi}{\partial\varphi} \right)$$

EXERCISE: calculate in spherical coordinates $\nabla \left(\frac{1}{r} \right)$

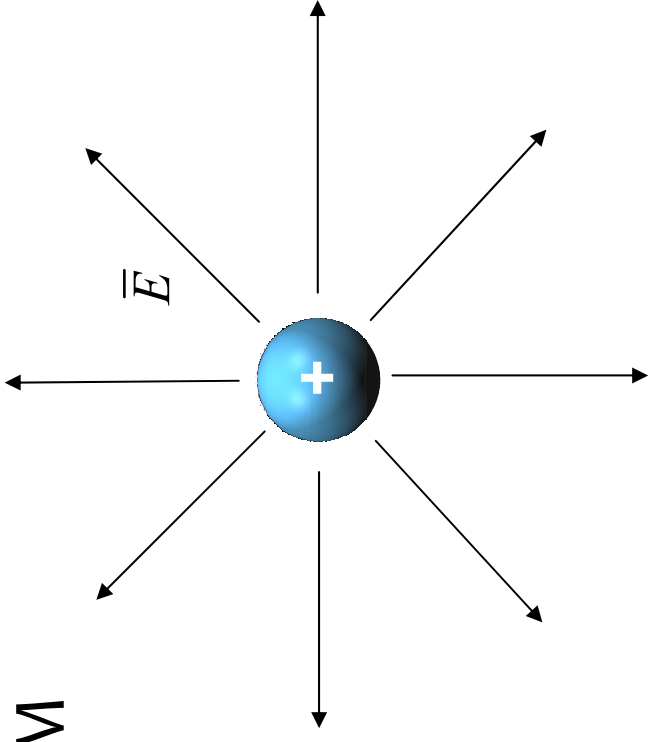
DIVERGENCE IN CURVILINEAR COORD.

$$\text{div}\vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right]$$

CURL IN CURVILINEAR COORD.

$$\text{rot}\vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

TARGET PROBLEM



$$\nabla^2 \phi = 0$$

Due to spherical symmetry

$$\phi = \phi(r)$$

$$\text{with } r = |\vec{r}|$$

Which can be written as:

$$\nabla^2 \phi = \nabla \cdot (\nabla \phi) = 0$$

Due to spherical symmetry, the solution is easy in spherical coordinates

$$\text{grad} \phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right)$$

$$\text{div} \bar{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right] = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (A_r r^2 \sin \theta) + \frac{\partial}{\partial \theta} (A_\theta r \sin \theta) + \frac{\partial}{\partial \varphi} (A_\varphi r \sin \theta) \right]$$

$$\text{div}(\text{grad} \phi) = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(\frac{\partial \phi}{\partial r} r^2 \sin \theta \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} r \sin \theta \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right) \right] = \frac{2}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2}$$

$\underbrace{\frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} r \sin \theta \right)}_{=0} + \underbrace{\frac{\partial}{\partial \varphi} \left(\frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right)}_{=0}$

$\phi = \phi(r) \Rightarrow$ No θ and no φ dependence

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} = 0 \Rightarrow \phi(r) = -\frac{c}{r} + d$$

WHICH STATEMENT IS WRONG?

- 1- The scale factor is necessary to calculate the gradient (yellow)
- 2- The scale factor is necessary to calculate the divergence (red)
- 3- Spherical coordinates are a curvilinear coordinate system (blue)
- 4- In spherical coordinates the position vector is $\vec{r} = (r, \theta, \varphi)$ (green)