VEKTORANALYS

Kursvecka 6

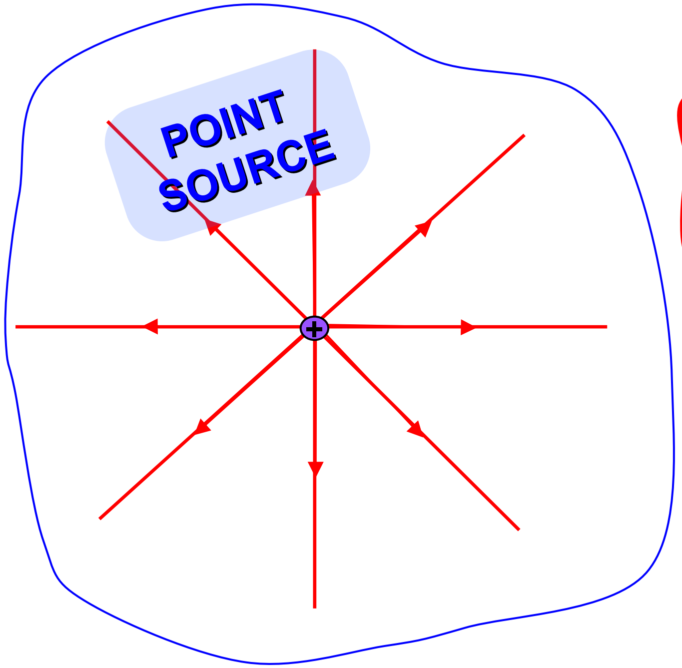
SOME SPECIAL VECTOR FIELDS
AND
LAPLACE AND POISSON EQUATIONS

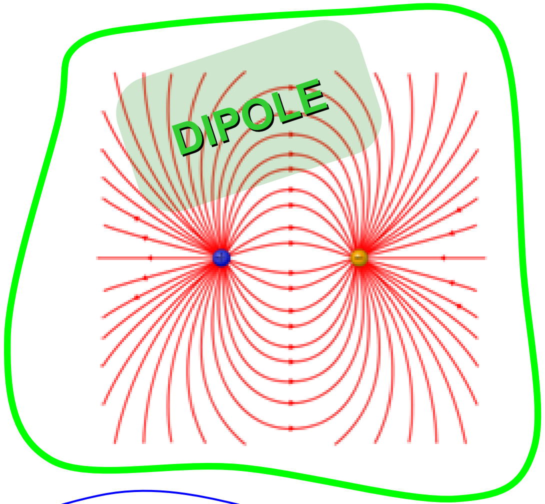
Kapitel 11-12
Sidor 123-150

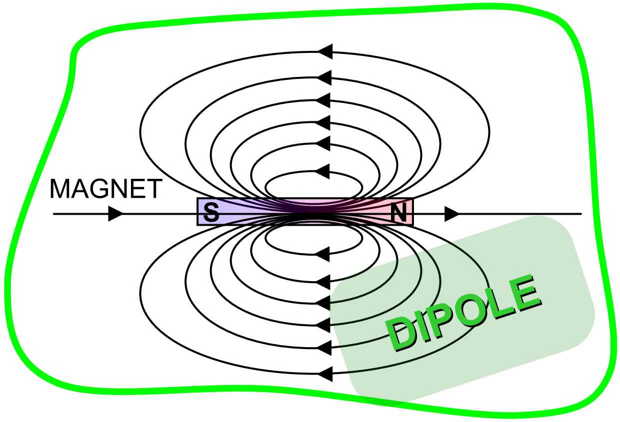
TARGET PROBLEM

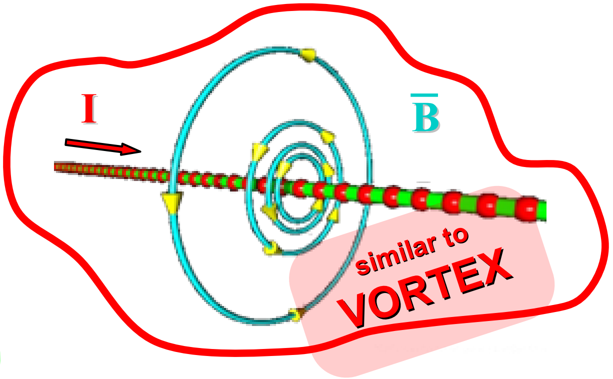
Which kind of sources for vector fields do we have in nature?

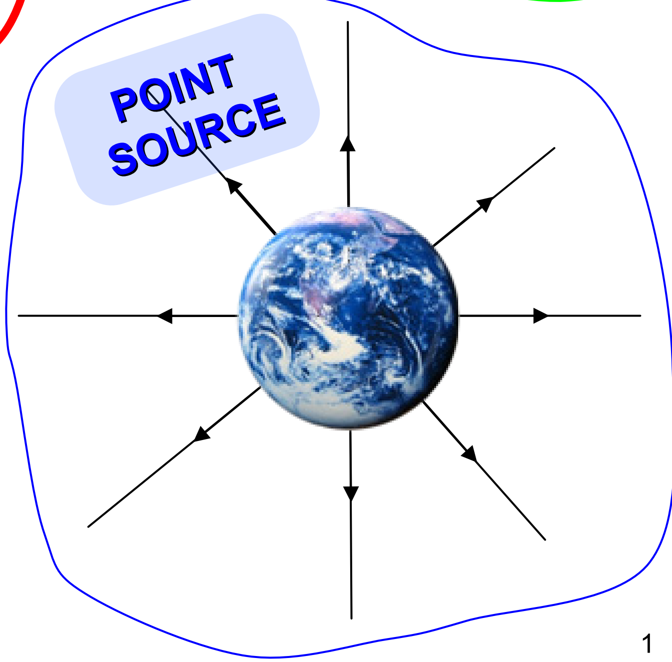
SOME EXAMPLES











TARGET PROBLEM

Characterization of different vector field sources:

- **Point source** *(punktkällan)*

  It is a single identifiable localized source with negligible extent.

  In some particular conditions,
  (for example: 3D space, emission homogenous in all directions, no absorption and no loss…)
  the field produced by the source decreases with r^2

- **Dipole source** *(dipolskällan)*

  Two identical but opposite sources (i.e. a source and a sink)
  separated by a distance $d \neq 0$.

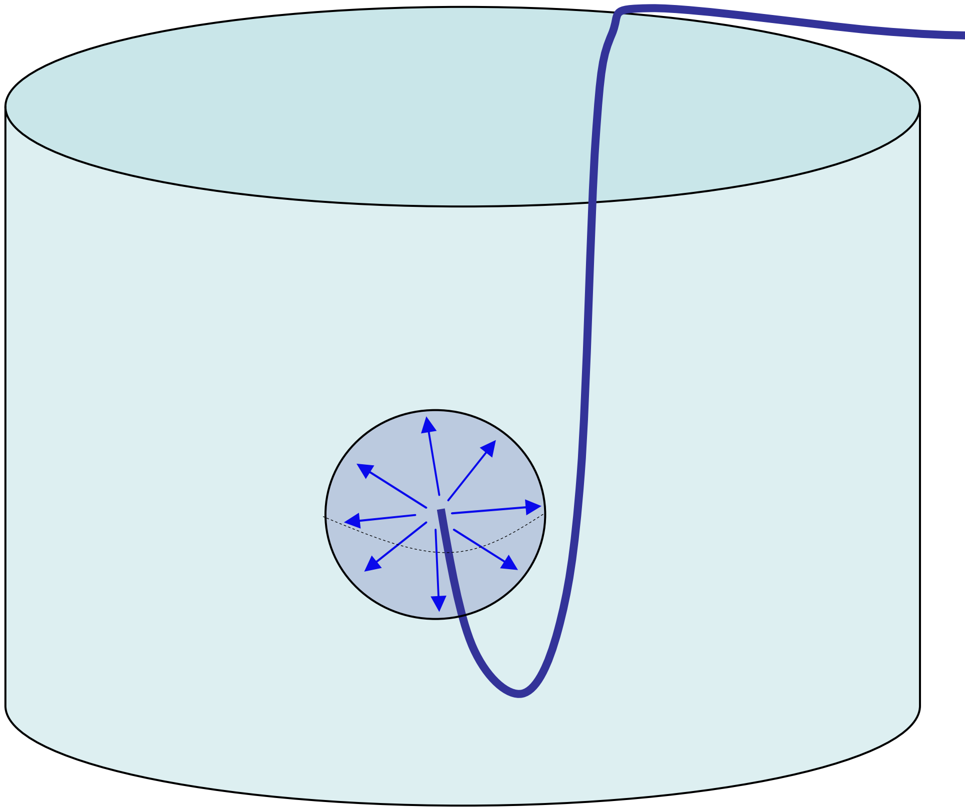
- **Vortex** *(virveltråden)*

  The velocity field in a water vortex
  Magnetic field around a straight wire
  …

POINT SOURCE

A single identifiable localized source with negligible extent.

Let's calculate the velocity field of the water that flows from a thin pipe into a large pool.



Assumptions

1- The source is homogeneous in time

the rate of the water from the pipe is constant: Volume/s=constant=F

2- The emission is homogeneous in all directions

3- No absorption, no losses

Then:

$$\left.\begin{array}{l} F = \overline{S} \cdot \overline{v} \\ \overline{S} = 4\pi r^2 \hat{e}_r \end{array}\right\} \Rightarrow \overline{v} = \frac{F}{4\pi r^2} \hat{e}_r$$

In general, the **vector field generated by a point source** is:

$$\boxed{\overline{A}(\overline{r}) = \frac{s}{r^2} \hat{e}_r}$$

POINT SOURCE

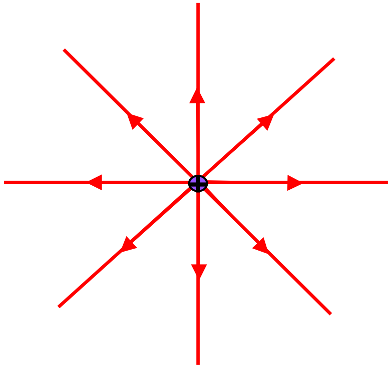
The vector field generated by a point source located in the origin is:

$$\overline{A}(\overline{r}) = \frac{s}{r^2}\hat{e}_r$$

If the source is not in the origin, then:

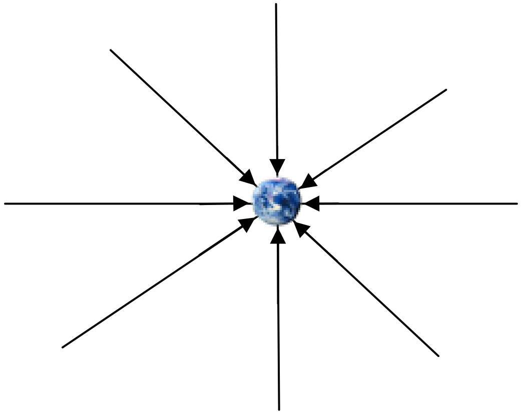
$$\overline{A}(\overline{r}) = s\frac{\overline{r}-\overline{r}_0}{\left|\overline{r}-\overline{r}_0\right|^3}$$

where $\overline{r}_0$ is the position of the source



- For the electrostatic field we have:

$$\overline{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{e}_r \qquad with \quad s = \frac{q}{4\pi\varepsilon_0}$$



- For the gravitational field we have:

$$\overline{g} = -GM\frac{1}{r^2}\hat{e}_r \qquad with \quad s = -GM$$

POINT SOURCE

THEOREM 1 *(11.1 in the book)*

The flux produced by a point source through a closed surface S (with S boundary of the volume V) is:

$$\oint\!\!\!\oint_S \frac{s}{r^2}\hat{e}_r \cdot d\overline{S} = \begin{cases} 0 & \textit{If the source is outside } V \\ 4\pi s & \textit{If the source is inside } V \end{cases}$$

PROOF

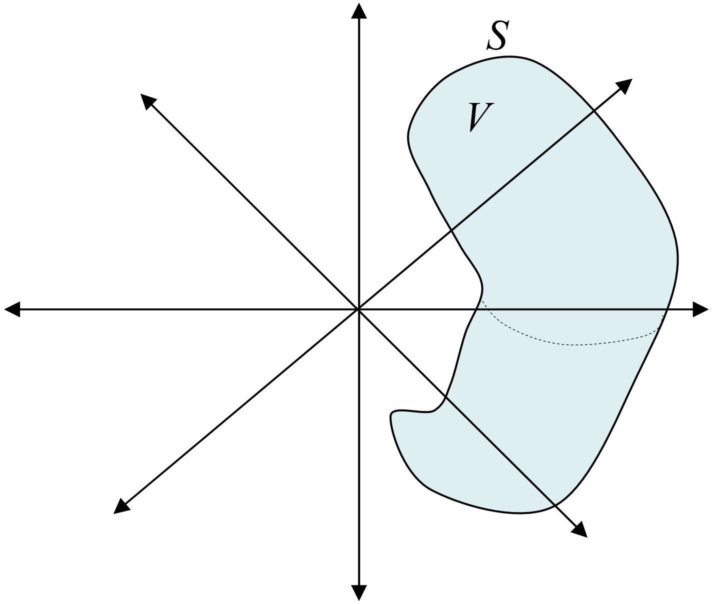
1. The origin is outside V

In V the field is continuously differentiable. Therefore we can apply the Gauss’ theorem:

$$\oint\!\!\!\oint_S \frac{s}{r^2}\hat{e}_r \cdot d\overline{S} = \iiint_V div\left(\frac{s}{r^2}\hat{e}_r\right)dV$$

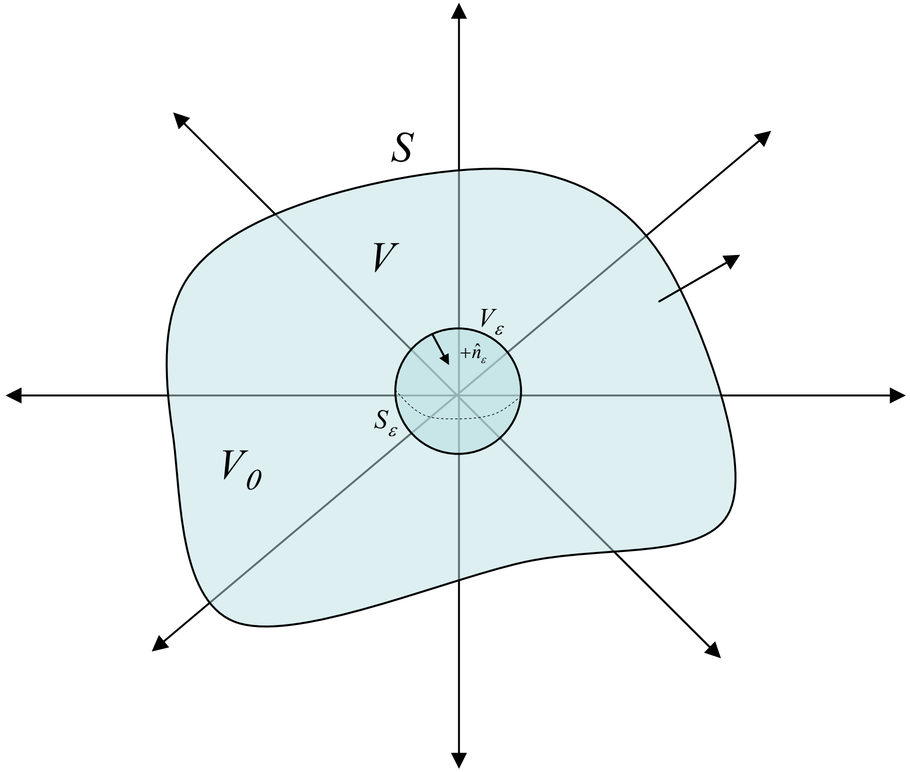
$$div\left(\frac{s}{r^2}\hat{e}_r\right) = \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{s}{r^2}\right) \qquad = 0$$

$$\Rightarrow \oint\!\!\!\oint_S \frac{s}{r^2}\hat{e}_r \cdot d\overline{S} = 0$$



2. The origin is inside V

In V the field is not continuous,
since the origin is a singular point!
Therefore we can NOT
apply the Gauss' theorem in V.



But we can divide V into two volumes:

$$V = V_0 + V_\varepsilon$$

V_ε is a "small" sphere with radius ε
with centre on the source (the origin).
V_0 is the remaining part of V

$$\iint_S \frac{s}{r^2}\hat{e}_r\cdot d\overline{S} = \iint_{S+S_\varepsilon-S_\varepsilon}\frac{s}{r^2}\hat{e}_r\cdot d\overline{S} =$$

$$\iint_{S+S_\varepsilon}\frac{s}{r^2}\hat{e}_r\cdot d\overline{S} + \iint_{-S_\varepsilon}\frac{s}{r^2}\hat{e}_r\cdot d\overline{S} =$$

Gauss'theorem: V_0 does not contain the origin

$$\iiint_{V_0}\underbrace{div\left(\frac{s}{r^2}\hat{e}_r\right)}_{=0}dV - \iint_{S_\varepsilon}\frac{s}{r^2}\hat{e}_r\cdot d\overline{S} = -\iint_{S_\varepsilon}\frac{s}{r^2}\hat{e}_r\cdot\underbrace{(-\hat{e}_r)}_{\hat{n}_\varepsilon=-\hat{e}_r}dS = \iint_{S_\varepsilon}\frac{s}{\varepsilon^2}dS = \frac{s}{\varepsilon^2}\iint_{S_\varepsilon}dS = \frac{s}{\varepsilon^2}\underbrace{4\pi\varepsilon^2}_{\text{Area of the sphere with radius }\varepsilon} = 4\pi s$$

THE POTENTIAL OF A POINT SOURCE

The potential from a point source is: $\boxed{\phi = -\frac{s}{r} + const.}$

In fact: $$grad\phi = \frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\underbrace{\frac{\partial \phi}{\partial \theta}}_{=0}\hat{e}_\theta + \frac{1}{r\sin\theta}\underbrace{\frac{\partial \phi}{\partial \varphi}}_{=0}\hat{e}_\varphi = -s\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{e}_r = \frac{s}{r^2}\hat{e}_r$$

ELECTROSTATIC FIELD FROM A POINT SOURCE

The electrostatic field from a point source is $\bar{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{e}_r$

In electrostatic the potential is often defined as $\bar{E} = -grad\phi_E$

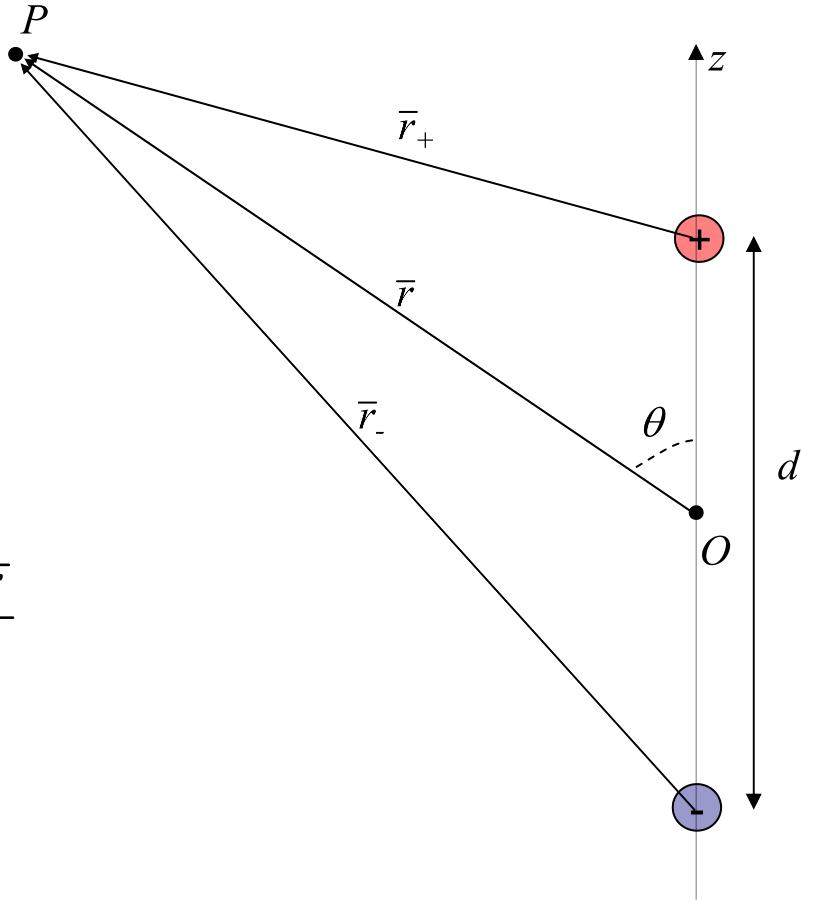
Therefore, the electrostatic potential is: $\boxed{\phi_E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$

The flux of the electric field is: $\boxed{\iint_S \bar{E}\cdot d\bar{S} = \frac{q}{\varepsilon_0}}$ *where q is the total charge inside S*

DIPOLE SOURCE

Two identical but opposite sources (i.e. a source and a sink) separated by a distant d.

The origin O is in the middle between the positive and the negative charge.



$$\begin{aligned} &If \quad r >> d \\ &r \approx r_+ \approx r_- \\ &r_- - r_+ \approx d\cos\theta \end{aligned}$$

The potential due to the dipole is:

$$\phi(\bar{r}) = \frac{s}{r_+} + \frac{-s}{r_-} = s\frac{r_- - r_+}{r_- r_+} \approx s\frac{d\cos\theta}{r^2} = s\frac{\bar{d}\cdot\bar{r}}{r^3}$$

Ideal dipole: $ds = constant$

The **dipole moment is defined as**: $\bar{p} \equiv s\bar{d}$

The field generated by the dipole is:

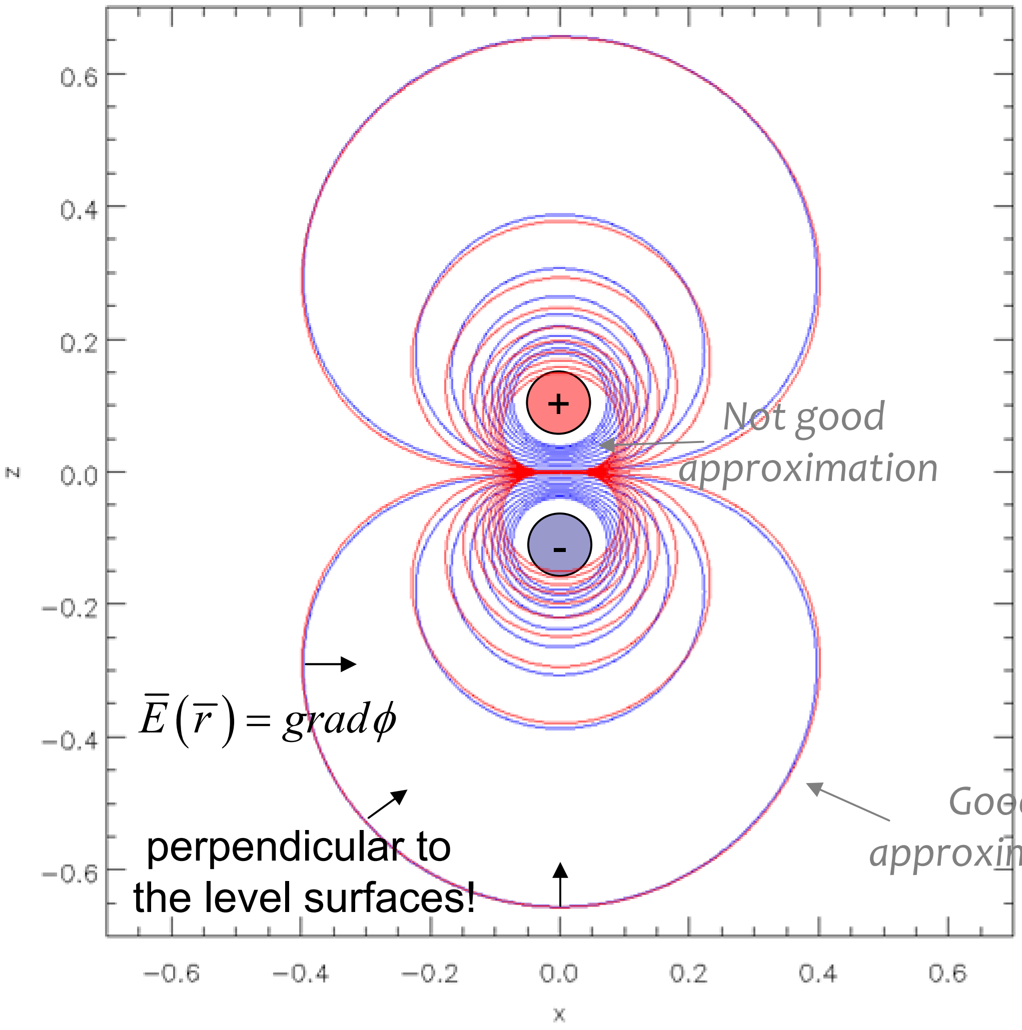
$$\bar{E}(\bar{r}) = -grad\phi = -grad\left(\frac{\bar{p}\cdot\bar{r}}{r^3}\right) = -\frac{\bar{p}}{r^3} + \frac{3(\bar{p}\cdot\bar{r})\bar{r}}{r^5}$$

$$\boxed{\begin{aligned} \phi(\bar{r}) &= \frac{\bar{p}\cdot\bar{r}}{r^3} \\ \bar{E}(\bar{r}) &= -\frac{\bar{p}}{r^3} + \frac{3(\bar{p}\cdot\bar{r})\bar{r}}{r^5} \end{aligned}}$$

DIPOLE SOURCE (example)

$$\phi(\bar{r}) = \frac{s}{r_+} - \frac{s}{r_-}$$

$$\phi(\bar{r}) = s\frac{d\cos\theta}{r^2}$$



Not good approximation

$$\bar{E}(\bar{r}) = grad\phi$$

perpendicular to the level surfaces!

Good approximation

VORTEX (or similar fields)

The velocity field in a water vortex, the magnetic field around a straight wire…

The vector field generated by a vortex has the shape: $\bar{A}(\bar{r}) = \frac{k}{\rho}\hat{e}_\varphi$

THEOREM 2 *(11.2 in the book)*

The circulation of this vector field is

$$\oint_L \frac{k}{\rho}\hat{e}_\varphi \cdot d\bar{r} = 2\pi k N$$

where N is number of turns of L around the z-axis

N is positive if the turn is along +L

N is negative if the turn is along -L

PROOF

The field has a singularity on the *z-axis*.

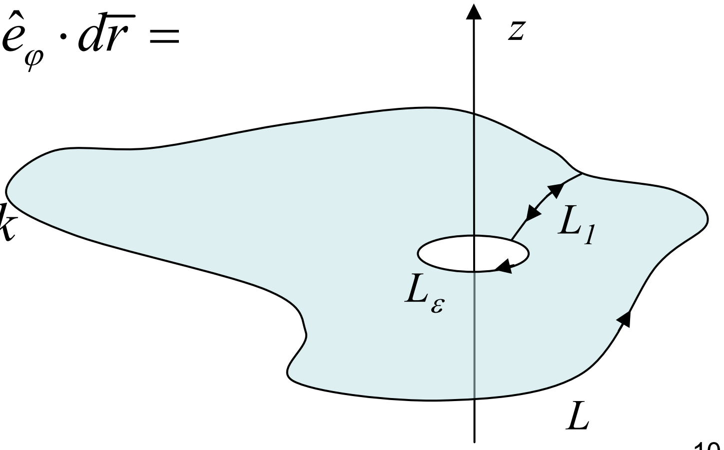
Therefore the Stokes' theorem cannot be applied directly.

We consider a circular path L_ε with radius ε

$$\int_L \frac{k}{\rho}\hat{e}_\varphi \cdot d\bar{r} = \int_{L+L_\varepsilon-L_\varepsilon+L_1-L_1} \frac{k}{\rho}\hat{e}_\varphi \cdot d\bar{r} = \underbrace{\int_{L+L_\varepsilon+L_1-L_1}}_{} \frac{k}{\rho}\hat{e}_\varphi \cdot d\bar{r} + \int_{-L_\varepsilon} \frac{k}{\rho}\hat{e}_\varphi \cdot d\bar{r} =$$

$$\iint_S rot\left(\frac{k}{\rho}\hat{e}_\varphi\right)\cdot d\bar{S} + \int_{-L_\varepsilon} \frac{k}{\rho}\hat{e}_\varphi \cdot d\bar{r} = \int_0^{2\pi} \frac{k}{\varepsilon}\underbrace{\varepsilon\hat{e}_\varphi \cdot \hat{e}_\varphi d\varphi}_{d\bar{r} = -\varepsilon\hat{e}_\varphi d\varphi} = 2\pi k$$

Closed path that does not contain the z-axis.
We can apply the Stokes' theorem!



WHICH STATEMENT IS WRONG?

1- The vector field $\frac{q}{r^2}\hat{e}_r$ is produced by a point source (yellow)

2- The vector field $\frac{k}{\rho}\hat{e}_\varphi$ can represent the velocity field of a vortex (red)

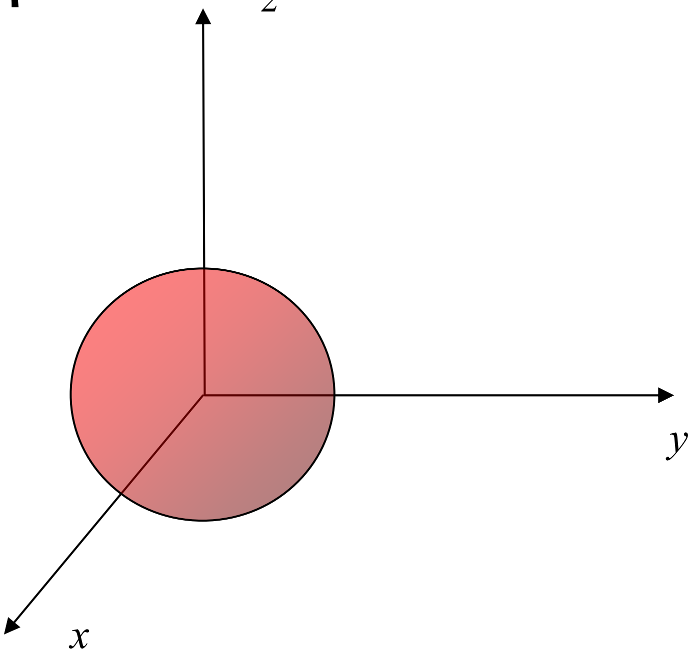
3- The flux of the field from a point source is always (green)

$$\iint_S \frac{s}{r^2}\hat{e}_r \cdot d\overline{S} = 4\pi s$$

4- The circulation $\int_L \frac{k}{\rho}\hat{e}_\varphi \cdot d\overline{r} = 2\pi k$ if L has only one turn around z (blue)

LAPLACE AND POISSON EQUATIONS

TARGET PROBLEM



A sphere has radius R and a charge density $\rho=\rho_c$.
Calculate:

- the electric field and
- the electrostatic potential

inside and outside the sphere.

We know:

$$\nabla \cdot \bar{E} = \frac{\rho_c}{\varepsilon_0}$$

$$\bar{E} = -\nabla \phi_E$$

Therefore: $$\nabla^2 \phi_E = -\frac{\rho_c}{\varepsilon_0}$$

This equation is an example of:

Laplace's equation $$\boxed{\nabla^2 \phi = 0}$$

Poisson's equation $$\boxed{\nabla^2 \phi = \rho}$$

SYMMETRIC SOLUTIONS

OF THE

LAPLACE EQUATION $\nabla^2\phi = 0$

PLANAR SYMMETRY $\phi = \phi(x)$ (NO *y* and *z* dependences)

In cartesian coord.

$$\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

$$\boxed{\frac{d^2\phi(x)}{dx^2} = 0} \Rightarrow \boxed{\phi(x) = ax + b}$$

CYLINDRICAL SYMMETRY $\phi = \phi(\rho)$ (NO φ and *z* dependences)

In cylindrical coord.

$$\nabla^2\phi = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

$$\boxed{\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\phi(\rho)}{d\rho}\right) = 0} \Rightarrow \rho\frac{d\phi(\rho)}{d\rho} = a$$

$$\Rightarrow \boxed{\phi(\rho) = a\ln\rho + b}$$

SPHERICAL SYMMETRY $\phi = \phi(r)$ (NO θ and φ dependences)

In spherical coord.

$$\nabla^2\phi = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}\right)$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi(r)}{dr}\right) = 0} \Rightarrow r^2\frac{d\phi(r)}{dr} = a$$

$$\boxed{\Rightarrow \quad \phi(r) = -\frac{a}{r} + b}$$

LAPLACE AND POISSON EQUATIONS

In general, it is not easy to solve these equations. However some theorems could help us.

THEOREM 1 *(12.2 in the book)*

If ϕ has continuous second derivatives in the volume V and $\phi = 0$ on the surface S that encloses V, then the solution to the Laplace equation $\nabla^2\phi = 0$ is:

$\phi\,(x,y,z)=0$ in V

PROOF

We know: $\nabla\cdot\left(f\,\bar{v}\right) = \left(\nabla f\right)\cdot\bar{v} + f\nabla\cdot\bar{v}$ **(ID2)**

$$\left.\begin{array}{l} f=\phi \\ \bar{v}=\nabla\phi \end{array}\right\} \Rightarrow \nabla\cdot(\phi\nabla\phi) = \nabla\phi\cdot\nabla\phi + \phi\left(\nabla\cdot\nabla\phi\right) = (\nabla\phi)^2 + \phi\underbrace{\nabla^2\phi}_{=0}$$

$$\Rightarrow \nabla\cdot(\phi\nabla\phi)-(\nabla\phi)^2 = 0 \quad \Rightarrow \iiint_V\left[\nabla\cdot(\phi\nabla\phi)-(\nabla\phi)^2\right]dV = 0$$

Gauss' theorem $\quad \|$

$$\underbrace{\iint_S \phi\nabla\phi\cdot d\bar{S}}_{=0} - \iiint_V \underbrace{(\nabla\phi)^2}_{\geq 0} dV = 0 \quad \Rightarrow \quad \phi = 0$$

because $\phi=0$ on S

DIRICHLET BOUNDARY CONDITIONS

$$\nabla^2\phi = \rho$$

$$\phi = \sigma \quad on\ S$$

Dirichlet boundary condition

What can we say about the solution?

THEOREM 2 *(12.3 in the book)*

The Poisson's equation $\nabla^2\phi = \rho$ in V
with boundary condition $\phi{=}\sigma$ on S
has only one solution.

PROOF Let's assume that ϕ_1 and ϕ_2 are two solution:

$$\nabla^2\phi_1 = \rho \quad and \quad \phi_1 = \sigma \quad on\ S$$

$$\nabla^2\phi_2 = \rho \quad and \quad \phi_2 = \sigma \quad on\ S$$

Let's now define $\phi_0{=}\phi_1{-}\phi_2$

$$\left.\begin{aligned} \nabla^2\phi_0 &= \nabla^2\left(\phi_1 - \phi_2\right) = \overbrace{\nabla^2\phi_1}^{\rho} - \overbrace{\nabla^2\phi_2}^{\rho} = 0 \\ \phi_0 &= \underbrace{\phi_1}_{\sigma} - \underbrace{\phi_2}_{\sigma} = 0 \quad on\ S \end{aligned}\right\}$$

Due to theorem 1: $\underbrace{\phi_0{=}0\ in\ V}$

$$\Downarrow$$

$\phi_1{=}\phi_2$ in V

NEUMANN BOUNDARY CONDITIONS

$$\nabla^2\phi = \rho$$

$$\frac{\partial\phi}{\partial n} = \hat{n}\cdot\nabla\phi = \gamma \quad on\ S$$

Neumann boundary condition

What can we say about the solution?

THEOREM 3 *(12.4 in the book)*

The solution to the Poisson's equation $\nabla^2\phi = \rho$ in V
with boundary condition $\hat{n}\cdot\nabla\phi = \gamma$ on S
is not unique. If ϕ is a solution then $\phi+c$ is also solution
where c is an arbitrary constant.

PROOF Let's assume that ϕ_1 and ϕ_2 are two solution:

$$\nabla^2\phi_1 = \rho \quad and \quad \hat{n}\cdot\nabla\phi_1 = \gamma \quad on\ S$$

$$\nabla^2\phi_2 = \rho \quad and \quad \hat{n}\cdot\nabla\phi_2 = \gamma \quad on\ S$$

Let's now define $\phi_0=\phi_1-\phi_2$

$$\left.\begin{aligned}\nabla^2\phi_0 &= \nabla^2\left(\phi_1-\phi_2\right) = \overbrace{\nabla^2\phi_1}^{\rho} - \overbrace{\nabla^2\phi_2}^{\rho} = 0 \\ \hat{n}\cdot\nabla\phi_0 &= \hat{n}\cdot\left(\underbrace{\nabla\phi_1}_{\gamma} - \underbrace{\nabla\phi_2}_{\gamma}\right) = 0 \quad on\ S\end{aligned}\right\} \Rightarrow \hat{n}\cdot\nabla\phi_0 = 0 \Rightarrow \phi_0\nabla\phi_0\cdot\hat{n} = 0 \ \ on\ S \Rightarrow \iint_S \phi_0\nabla\phi_0\cdot\hat{n}dS = 0$$

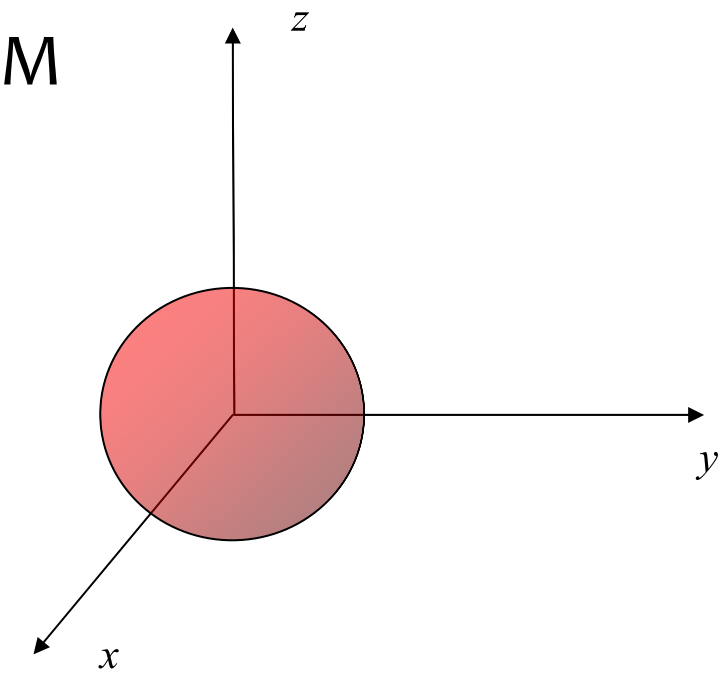
$$0 = \iint_S \phi_0\nabla\phi_0\cdot\hat{n}dS \underset{\text{Gauss' theorem}}{=} \iiint_V \nabla\cdot\phi_0\nabla\phi_0 dV \underset{\text{see proof of theorem 1}}{=} \iiint_V \underbrace{\left(\nabla\phi_0\right)^2}_{\geq 0} dV \quad \Rightarrow \quad \nabla\phi_0 = 0 \quad \Rightarrow \quad \phi_0 = const.$$

$$\Rightarrow \quad \phi_1 = \phi_2 + const.$$

TARGET PROBLEM

A sphere has radius R and charge density $\rho=\rho_c$.
Calculate:
- the electric field and
- the electrostatic potential

inside and outside the sphere.



Spherical symmetry: $\phi = \phi(r)$

Outside the sphere

$$\nabla^2 \phi_E = 0 \quad \Rightarrow \quad \phi_E^{out}(r) = -\frac{a}{r} + b$$

typically

$$\lim_{r\to\infty} \phi_E(r) = 0 \quad \Rightarrow \quad b = 0$$

$$\bar{E} = -\nabla \phi_E = -\left(\frac{d\phi_E(r)}{dr}, \frac{1}{r}\frac{d\phi_E(r)}{d\theta}, \frac{1}{r\sin\theta}\frac{d\phi_E(r)}{d\varphi} \right) \quad \Rightarrow E_r^{out} = -\frac{d\phi_E^{out}(r)}{dr} = -\frac{a}{r^2}$$

Inside the sphere

$$\nabla^2 \phi_E = -\frac{\rho_c}{\varepsilon_0} \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi_E(r)}{dr} \right) = -\frac{\rho_c}{\varepsilon_0}$$

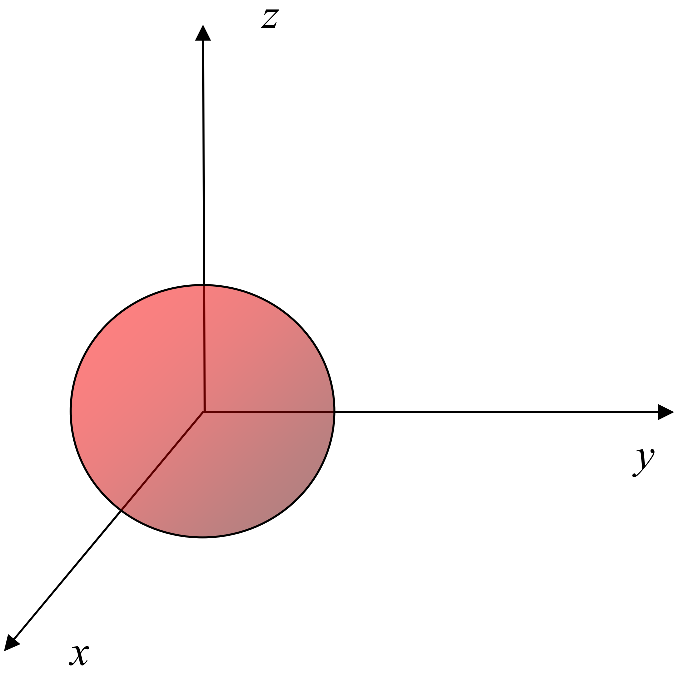
multiplying by r² and integrating:

$$r^2 \frac{d\phi_E(r)}{dr} = -\frac{\rho_c r^3}{3\varepsilon_0} + c \;\Rightarrow\; \frac{d\phi_E(r)}{dr} = -\frac{\rho_c r}{3\varepsilon_0} + \frac{c}{r^2} \quad \Rightarrow \; \phi_E^{in}(r) = -\frac{\rho_c r^2}{6\varepsilon_0} + d$$

$$E_r^{in} = -\frac{d\phi_E^{in}(r)}{dr} = +\frac{\rho_c r}{3\varepsilon_0} - \frac{c}{r^2}$$

Divergent at r=0
NOT physical! $\Rightarrow$ *c=0*

TARGET PROBLEM



We still have to calculate a and d!

Boundary conditions:

$$E_r^{out}(R)=E_r^{in}(R) \quad \Rightarrow \quad -\frac{a}{R^2}=\frac{\rho_c R}{3\varepsilon_0} \quad \Rightarrow \quad a=-\frac{\rho_c R^3}{3\varepsilon_0}$$

$$\phi_E^{out}(R)=\phi_E^{in}(R) \quad \Rightarrow -\frac{\rho_c R^2}{6\varepsilon_0}+d=\frac{\rho_c R^3}{3\varepsilon_0 R} \quad \Rightarrow \quad d=\frac{\rho_c R^2}{2\varepsilon_0}$$

$$\boxed{\phi_E^{out}(r)=\frac{\rho_c R^3}{3\varepsilon_0 r}} \qquad \boxed{E_r^{out}=+\frac{\rho_c R^3}{3\varepsilon_0 r^2}}$$

$$\boxed{\phi_E^{in}(r)=\frac{\rho_c R^2}{6\varepsilon_0}\left(3-\frac{r^2}{R^2}\right)} \qquad \boxed{E_r^{in}=+\frac{\rho_c r}{3\varepsilon_0}}$$