VEKTORANALYS

Kursvecka 5

övningar

PROBLEM 1

(a) Apply the formal "nablaräkning" to *rot rot A* to obtain an expression for $\nabla^2 \overline{A}$

(b) Use the previous expression to calculate $\nabla^2 \hat{e}_r$ in spherical coord.

SOLUTION

(a) $\quad \nabla \times \left(\nabla \times \overline{A} \right) = \nabla \times \left(\nabla \times \overline{A} \right) = \nabla \left(\nabla \cdot \overline{A} \right) - \left(\nabla \cdot \nabla \right) \overline{A} = \nabla \left(\nabla \cdot \overline{A} \right) - \nabla^2 \overline{A}$

$$\uparrow$$

$$\overline{n} \times \left(\overline{n} \times \overline{c} \right) = \overline{n} \left(\overline{n} \cdot \overline{c} \right) - \overline{c} \left(\overline{n} \cdot \overline{n} \right)$$

Therefore: $\quad \nabla^2 \overline{A} = \nabla \left(\nabla \cdot \overline{A} \right) - \nabla \times \left(\nabla \times \overline{A} \right)$

(b) $\quad \nabla^2 \hat{e}_r = \nabla \left(\nabla \cdot \hat{e}_r \right) - \nabla \times \left(\nabla \times \hat{e}_r \right)$

We need gradient, divergence and curl in spherical coordinates

1

SOLUTION

Spherical coord: (r, θ, φ)

$$\overline{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$$

$$\begin{cases} h_r = 1 \\ h_\theta = r \\ h_\varphi = r\sin\theta \end{cases}$$

$$grad\,\phi = \left(\frac{1}{h_1}\frac{\partial\phi}{\partial u_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial u_3} \right) = \left(\frac{1}{h_r}\frac{\partial\phi}{\partial r}, \frac{1}{h_\theta}\frac{\partial\phi}{\partial\theta}, \frac{1}{h_\varphi}\frac{\partial\phi}{\partial\varphi} \right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi} \right)$$

$$\boxed{grad\,\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi} \right)}$$

$$div\,\overline{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2) \right] =$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(A_r r^2\sin\theta) + \frac{\partial}{\partial\theta}(A_\theta r\sin\theta) + \frac{\partial}{\partial\varphi}(A_\varphi r) \right]$$

$$\boxed{div\,\overline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(A_\varphi)}$$

2

SOLUTION

$$
rot\overline{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta \hat{e}_\varphi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}
$$

$$
rot\overline{A} = \left(\frac{1}{r\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta A_\varphi\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}, \; \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right), \; \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta} \right)
$$

Let's go back to our initial problem:

$$
\nabla^2 \hat{e}_r = grad\left(div\,\hat{e}_r\right) - rot\left(rot\,\hat{e}_r\right)
$$

$$
\hat{e}_r = (1,0,0) \qquad \textit{(in the spherical coordinate system!)}
$$

So we have to calculate:

$$
div\overline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(A_\varphi\right)
$$

$$
\textit{with } \overline{A} = \hat{e}_r = (1,0,0)
$$

3

SOLUTION

Therefore: $\quad div\, \hat{e}_r = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\right) = \dfrac{2}{r}$

Then we have to calculate:

$$rot\overline{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\varphi\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}, \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right), \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)$$

$with\ \overline{A} = \hat{e}_r = (1,0,0)$

Therefore: $\quad rot\,\hat{e}_r = (0,0,0)$

$$grad\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$$

Finally:

$$\nabla^2\hat{e}_r = grad\left(div\,\hat{e}_r\right) - rot\left(rot\,\hat{e}_r\right) = grad\left(\frac{2}{r}\right) - \nabla\times(0,0,0) = \left(-\frac{2}{r^2},0,0\right) = -\frac{2}{r^2}\hat{e}_r$$

4

PROBLEM 2

Consider the following fields: $\quad \psi = -\dfrac{\cos\theta}{r^2}, \qquad \bar{A} = \dfrac{\sin\theta}{r^2}\hat{e}_\varphi$

where (r,θ,φ) are spherical coordinates.
Calculate:

a) $\quad \nabla\psi$

b) $\quad \nabla\times\bar{A}$

c) $\quad \nabla^2\psi \quad and \quad \nabla\times\left(\nabla\times\bar{A}\right)$

SOLUTION

(a) $\quad grad\psi = \left(\dfrac{\partial\psi}{\partial r}, \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}, \dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial\varphi}\right) = \left(\dfrac{2\cos\theta}{r^3}, \dfrac{\sin\theta}{r^3}, 0\right) = \dfrac{2\cos\theta}{r^3}\hat{e}_r + \dfrac{\sin\theta}{r^3}\hat{e}_\theta$

(b) $\quad rot\bar{A} = \left(\dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta A_\varphi\right) - \dfrac{1}{r\sin\theta}\dfrac{\partial A_\theta}{\partial\varphi}, \dfrac{1}{r\sin\theta}\dfrac{\partial A_r}{\partial\varphi} - \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(rA_\varphi\right), \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(rA_\theta\right) - \dfrac{1}{r}\dfrac{\partial A_r}{\partial\theta}\right)$

$\quad with\ \bar{A} = \left(0, 0, \dfrac{\sin\theta}{r^2}\right)$

5

SOLUTION

$$rot\overline{A} = \left(\frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left(\frac{\sin^2\theta}{r^2} \right), -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r} \right), 0 \right) = \left(\frac{1}{r^3\sin\theta} 2\sin\theta\cos\theta, \frac{\sin\theta}{r}\frac{1}{r^2} \right) =$$

$$\left(\frac{2\cos\theta}{r^3}, \frac{\sin\theta}{r^3}, 0 \right) = grad\psi$$

(c) $$\nabla^2\psi = \nabla\cdot(\nabla\psi) = \nabla\cdot(\nabla\times\overline{A}) = 0$$

$$\nabla\times(\nabla\times\overline{A}) = \nabla\times(\nabla\psi) = 0$$

see ID7 and ID8 in lesson 4

6

PROBLEM 3

The Orion nebula can be considered as a sphere with constant radius R_0.
Assume that the velocity of the Helium atoms in the nebula is described by the following field:

$$\bar{v} = k\left(r - \frac{r^2}{2}\cos\varphi \right)\hat{e}_r + k\left(2r^2 \sin\theta \sin\varphi \right)\hat{e}_\varphi$$

Calculate the flux of Helium atoms that flows outside the nebula

SOLUTION

$$Flux = \iint_S \bar{v}\cdot d\bar{S} = \iiint_V \overset{\text{Guass theorem}}{div(\bar{v})}dV$$

$$\nabla\cdot\bar{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(v_\varphi\right) \qquad \text{(in a spherical coord. system)}$$

$$div\left(k\left(r - \frac{r^2}{2}\cos\varphi \right), 0, 2r^2 k\sin\theta\sin\varphi \right) = k\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 - \frac{r^4}{2}\cos\varphi \right) + k\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(2r^2\sin\theta\sin\varphi \right) =$$

$$= \frac{k}{r^2}\left(3r^2 - 4\frac{r^3}{2}\cos\varphi \right) + k_2\frac{2r^2\sin\theta}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\sin\varphi\right) = 3k - 2kr\cos\varphi + 2kr\cos\varphi = 3k$$

$$\iint_S \bar{v}\cdot d\bar{S} = \iiint_V 3k\,dV = 3\frac{4\pi}{3}R_0^3 = 4\pi kR_0^3$$