PhD Course in KTH - Sparse Signal Processing
Slides 4
Discussion Topic - Pursuit Algos' Guarantees

Saikat Chatterjee

Communication Theory Lab, KTH

March 11, 2014

The Guarantee Question

Remark

Question: Assume that $\mathbf{Ax} = \mathbf{b}$ *has a sparse solution with* k_0 *non-zeros, i.e.,* $\|\mathbf{x}\|_0 = k_0$*. Furthermore, assume that* $k_0 < \frac{1}{2} spark(\mathbf{A})$*. Will OMP, BP succeed in recovering the sparsest solution?*

1. Note that, such success for any k_0 and $\mathbf{A}$ is not possible due to the known conflict of NP-hardness.
2. However, if the solution is “sufficiently sparse”, the success of the some algos is guaranteed.

OMP Performance Guarantee

Theorem

Equivalence - OMP - Two-Ortho : *For* $\mathbf{A}\mathbf{x} = [\mathbf{\Psi}\ \mathbf{\Phi}]\,\mathbf{x} = \mathbf{b}$, *if a solution* $\mathbf{x}$ *exists such that it has* k_p *non-zeros in its first half and* k_q *non-zeros in its second half, and the two obey*

$$max(k_p, k_q) < \frac{1}{2\mu\left(\mathbf{A}\right)}, \tag{1}$$

OMP is guaranteed to find the solution in $k_0 = k_p + k_q$ *steps.*

Proof: We work out.

BP Performance Guarantee

Theorem

Equivalence - BP - Two-Ortho : For $\mathbf{A}\mathbf{x} = [\mathbf{\Psi} \;\; \mathbf{\Phi}]\,\mathbf{x} = \mathbf{b}$, if a solution $\mathbf{x}$ exists such that it has k_p non-zeros in its first half and $k_q \leq k_p$ non-zeros in its second half, and the two obey

$$2\mu\left(\mathbf{A}\right)^2 \; k_p \, k_q + \mu\left(\mathbf{A}\right) k_p - 1 < 0, \qquad (2)$$

then that solution is both the unique solution of (P_1) and the unique solution of (P_0).

1. The above condition is difficult to interpret. So, instead, we provide a weaker condition for equivalence between the solutions of (P_1) and (P_0).
2. The weak condition: $\|\mathbf{x}\|_0 = k_0 = k_p + k_q < \frac{\sqrt{2}-0.5}{\mu(\mathbf{A})}$.

Proof: More involving (Home Work).

An illustrative example

Question: How the bounds look like? Any idea?
Let us discuss by some illustrations

OMP Performance Guarantee

Theorem

Equivalence - OMP : For $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is full row rank, if a solution $\mathbf{x}$ exists such that

$$\|\mathbf{x}\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(\mathbf{A})}\right), \tag{3}$$

OMP is guaranteed to find the solution.

Proof: We work out.

BP Performance Guarantee

Theorem

Equivalence - BP : For $\mathbf{A}\mathbf{x} = \mathbf{b}$*, if a solution* $\mathbf{x}$ *exists such that*

$$\|\mathbf{x}\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(\mathbf{A})}\right), \tag{4}$$

then that solution is both the unique solution of (P_1) *and the unique solution of* (P_0)*.*

Proof: We work out.

1. Note: In general case, both OMP and BP have same worst case bounds. This is not alright.
2. Can we do something more? Different kind of tools and analysis? Yes, we can, like the Tropp's Exact Recovery Condition. But, we mostly skip them. More interested readers should go by themselves.