

## LECTURE 2:

### EIGENVALUES, EIGENVECTORS AND SIMILARITY



“The single most important concept in matrix theory.”

German word “eigen” means *proper* or *characteristic*.

### EXAMPLE: STABILITY OF LINEAR SYSTEMS

Consider a linear discrete time homogenous system:

$$x(n+1) = Ax(n)$$

If the spectral radius of  $A$  is greater than 1, and  $x(n)$  is not orthogonal to the corresponding eigenvectors,  $x(n+1)$  will grow in the direction of the unstable modes.



## DEFINITION

**Consider:** Square matrix  $A \in M_n$

If there exists  $x \in \mathbf{C}^n$  ( $x \neq 0$ ) and  $\lambda \in \mathbf{C}$  such that

$$Ax = \lambda x$$

then

- $\lambda$  is an eigenvalue of  $A$
- $x$  is an eigenvector of  $A$  associated with  $\lambda$



### More terminology:

- Spectrum of  $A$ : Set of all eigenvalues. Notation:  $\sigma(A)$ .
- Eigenspace of  $A$ : Set of all eigenvectors.
- Spectral radius of  $A$ :  $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ .

### EXAMPLE: FILTER OUTPUT POWER MAXIMIZATION

Let  $y(t)$  be a vector of observations at time  $t$ , and let  $z(t) = x^T y(t)$  be the output of a filter with filter coefficients in the vector  $x$ . The mean output power is

$$\mathbf{E}\{z^2(t)\} = \mathbf{E}\{(x^T y(t))(y^T(t)x)\} = x^T \mathbf{E}\{y(t)y^T(t)\}x = x^T R_y x$$

Assume we want to find the filter that maximizes the output power under a normalizing constraint on the filter gain

$$\max_x x^T R_y x \quad \text{subject to } x^T x = 1$$

The solution to this is given by the unit length eigenvector corresponding to the largest eigenvalue of  $R_y$ .



## WHAT ABOUT ZERO?

**Recall:** If exists  $x \in \mathbf{C}^n$  ( $x \neq 0$ ) and  $\lambda \in \mathbf{C}$  such that

$$Ax = \lambda x$$

then

- $\lambda$  is an eigenvalue of  $A$
- $x$  is an eigenvector of  $A$  associated with  $\lambda$

**Question:** Why  $x \neq 0$ ?

**Answer:** We always have  $A0 = \lambda 0$  (uninteresting solution!)

**However:**  $\lambda = 0$  is an important case:  $Ax = 0 = 0x$

**In fact:**  $A \in M_n$  is singular iff it exists  $x \neq 0$  such that  $Ax = 0x = 0$  or, equivalently, iff  $0 \in \sigma(A)$



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## HOW TO FIND AN EIGENVALUE?

Rewrite eigenvalue definition ( $A \in M_n$ ):

$$\lambda x - Ax = 0 \Leftrightarrow (\lambda I - A)x = 0$$

**Observation:** Eigenvalues make  $(\lambda I - A)$  singular,  $\det(\lambda I - A) = 0$ .

**Definition:** Characteristic polynomial is  $p_A(t) = \det(tI - A)$ .

$p_A(t)$  is polynomial of degree  $n$ : Has  $n$  solutions/roots to  $p_A(t) = 0$ .

**Conclusions:** These roots are the eigenvalues of  $A$ .  
 $A \in M_n$  has  $n$  (complex) eigenvalues.

Some eigenvalues may have (algebraic) multiplicity!



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## POLYNOMIALS OF MATRICES

**Consider:** Scalar polynomial  $p(t) = a_k t^k + a_{k-1} t^{k-1} + \dots + a_0$ .

**Define:** Matrix polynomial  $p(A) = a_k A^k + a_{k-1} A^{k-1} + \dots + a_0 I$  for  $A \in M_n$ .



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**Theorem:** If  $\lambda$  is an eigenvalue of  $A$  and  $x$  the associated eigenvector, then

$$p(A)x = p(\lambda)x.$$

Thus,  $x$  is also eigenvector of  $p(A)$  associated with eigenvalue  $p(\lambda)$ .

## TRACE AND DETERMINANT

**Definition:** Trace is  $\text{tr}(A) = \sum_{i=1}^n a_{ii}$

**Definition:** Determinant is  $\det(A) = [\text{Laplace expansion in 0.3.1}]$

**Theorem:** Expressed using eigenvalues as

$$\text{tr}(A) = \sum_{i=1}^n \lambda_i, \quad \det(A) = \prod_{i=1}^n \lambda_i.$$

**Observation:** Coefficients in characteristic polynomial

$$p_A(t) = t^n + a_{n-1} t^{n-1} + \dots + a_0$$

where  $a_{n-1} = -\text{tr}(A)$ ,  $a_0 = (-1)^n \det(A)$ .

Formulas exist for all  $a_k$ ; see the book.

## SIMILARITY

**Consider:**  $A \in M_n, B \in M_n$

**Definition:**  $B$  is similar to  $A$  if there exists a nonsingular  $S \in M_n$  such that

$$A = S^{-1}BS$$

**Notation:**  $B \sim A$

Transformation  $A \rightarrow S^{-1}AS$  called *similarity transformation* by the similarity matrix  $S$ .



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## EQUIVALENCE CLASS

Similarity is

**reflexive**  $A \sim A$

**symmetric**  $B \sim A$  implies  $A \sim B$

**transitive**  $C \sim B$  and  $B \sim A$  imply  $C \sim A$

Divides all matrices into (disjoint) equivalence classes:

- Each class has a representative matrix  $A$ .
- The class includes all matrices similar to  $A$ .

**Theorem:** If  $B \sim A$ , then  $p_B(t) = p_A(t)$ .

$B$  and  $A$  have the same eigenvalues (counting multiplicity).



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## DIAGONALIZABLE MATRICES

**Definition:**

$A \in M_n$  is diagonalizable if it is similar to a diagonal matrix.

**Theorem:**

$A$  is diagonalizable iff  $A$  has  $n$  linearly independent eigenvectors.



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## LINEARLY INDEPENDENT EIGENVECTORS

**Assume:**

$\lambda_1, \lambda_2, \dots, \lambda_n$  are distinct eigenvalues of  $A \in M_n$

$x_i$  is eigenvector associated with  $\lambda_i$

**Theorem:**  $\{x_1, x_2, \dots, x_n\}$  is a linearly independent set.

**Conclusion:** If  $A \in M_n$  has  $n$  distinct eigenvalues then  $A$  is diagonalizable.

(The converse is not true.)



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**Definition:**  $A, B \in M_n$  commute if  $AB = BA$

**Definition:** Two diagonalizable matrices  $A, B \in M_n$  are *simultaneously diagonalizable* if there exists a single similarity matrix  $S \in M_n$  diagonalizing both  $A$  and  $B$ .

**Theorem:**  $A, B$  commute iff they are simultaneously diagonalizable.



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## HOW TO FIND THE EIGENVECTOR?

Rewrite eigenvalue definition ( $A \in M_n$ ):

$$\lambda x - Ax = 0 \quad \Leftrightarrow \quad (\lambda I - A)x = 0$$

**Observation:**  $x$  lies in the nullspace of  $\lambda I - A$ .

**Calculate eigenvectors:** Solve  $(\lambda I - A)x = 0$  for eigenvalue  $\lambda$ .

System of equations: Use Gauss elimination.

Eigenvector is non-unique:

- Any scaling ( $x \neq 0$ )
- Nullspace can have large dimension.



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**Assume:**  $A \in M_{m,n}$  and  $B \in M_{n,m}$  with  $m \leq n$ .

**Theorem:**  $p_{BA}(t) = t^{n-m} p_{AB}(t)$

$BA$  has the same eigenvalues as  $AB$  plus  $n - m$  additional eigenvalues at zero.

If  $m = n$  and  $A$  (or  $B$ ) is nonsingular, then  $AB$  is similar to  $BA$ .



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## EIGENSPACE

The set of all eigenvectors satisfying  $Ax = \lambda x$  for a given  $\lambda \in \sigma(A)$  is called the eigenspace of  $A$  corresponding to  $\lambda$ .

The eigenspace, together with the zero vector, is a subspace of  $\mathbf{C}^n$  and it is exactly the nullspace of  $\lambda I - A$ .



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## MULTIPLICITY

**Algebraic multiplicity:** Multiplicity of the corresponding root of the characteristic polynomial.

**Geometric multiplicity:** Number of linearly independent eigenvectors associated with the eigenvalue.



**Theorem:** Algebraic multiplicity  $\geq$  Geometric multiplicity

**Definition:**

If strict inequality for some eigenvalue, the matrix is *defective*.

**Theorem:**  $A$  is diagonalizable iff it is not defective.

## TRANSPOSE AND CONJUGATE TRANSPOSE

**Transpose:**

- $A$  and  $A^T$  have same eigenvalues.
- Left/right eigenvectors are interchanged and complex conjugated.



**Conjugate transpose:**

- Eigenvalues of  $A^*$  are complex conjugates of eigenvalues of  $A$ .
- Left/right eigenvectors are interchanged.

## LEFT EIGENVECTORS

A nonzero vector  $y \in \mathbb{C}^n$  is a left eigenvector of  $A \in M_n$  if

$$y^*A = \mu y^*.$$

Observe that  $\mu \in \sigma(A)$ .



**Theorem (Biorthogonality):** Let

$$y^*A = \mu y^* \quad \text{and} \quad Ax = \lambda x.$$

Then, if  $\mu \neq \lambda$  we have  $y^*x = 0$ .

**Observation:**

If  $A$  is Hermitian ( $A = A^*$ ), then  $x = y$  for same eigenvalue. Biorthogonality then implies that  $A$  has  $n$  pair-wise orthogonal eigenvectors of (at least if eigenvalues are distinct, more later).