LECTURE 3: OUTLINE

- Ch. 2: Unitary equiv, QR factorization, Schur's thm, Cayley-H., Normal matrices, Spectral thm, Singular value decomp.
- Ch. 3: Canonical forms: Jordan/Matrix factorizations



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UNITARY MATRICES

- A set of vectors $\{x_i\} \in {f C}^n$ are called
- orthogonal if $x_i^* x_j = 0, \forall i \neq j$ and
- orthonormal if they are orthogonal and $x_i^* x_i = 1, \ \forall i.$
- A matrix $U \in M_n$ is *unitary* if $U^*U = I$.
- A matrix $U \in M_n(\mathbf{R})$ is real orthogonal if $U^T U = I$.
- (A matrix $U \in M_n$ is orthogonal if $UU^T = I$.)
- If $U\!,V$ are unitary then UV is unitary.
 - Unitary matrices form a group under multiplication.

The following are equiv.

- 1. U is unitary
- 2. U is nonsingular and $U^{-1} = U^*$
- 3. $UU^* = I$

4. U^* is unitary

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- 5. the columns of U are orthonormal
- 6. the rows of U are orthonormal
- 7. for all $x \in \mathbf{C}^n$, the Euclidean length of y = Ux equals that of x.
- Def: A linear transformation $T : \mathbf{C}^n \to \mathbf{C}^m$ is a *Euclidean isometry* if $x^*x = (Tx)^*(Tx)$ for all $x \in \mathbf{C}^n$
- $x = (1x)^{*}(1x)$ for all $x \in \mathbb{C}^{n}$
- Unitary \boldsymbol{U} is an Euclidean isometry.

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EUCLIDEAN ISOMETRY AND PARSEVAL'S THEOREM

1. F_N be the FFT (Fast Fourier Transform matrix) of dimension N imes N, i. e,

$$F_N(m,n) = \frac{1}{\sqrt{N}}e^{\frac{-2\pi(m-1)(n-1)}{N}}$$

2. F is a unitary matrix.



- 3. Let $y = F_N x$ i.e, y is the N point FFT of x.
- (a) Length of x = Length of y
- (b) $\sum_{j=1}^N |x(j)|^2 = \sum_{j=1}^N |y(j)|^2$: This is energy conservation or Parseval's Theorem in DSP.

UNITARY EQUIVALENCE

Def: A matrix $B\in M_n$ is unitarily equivalent (or similar) to $A\in M_n$ if $B=U^*AU$ for some unitary matrix U.

(i) $A \rightarrow S^{-1}AS$: similarity (Ch 1,3)



(ii) $A \to S^*AS$: *congruence (Ch 4)

(iii) $A \to S^*AS$ with S unitary : unitary similarity (Ch 2)

Since in (iii) $S^{\ast}=S^{-1},$ we have that (iii) is "included" in both (i) and (ii).

Theorem: If ${\cal A}$ and ${\cal B}$ are unitarily equivalent then

$$A\|_{F}^{2} \triangleq \sum_{i,j} |a_{ij}|^{2} = \sum_{i,j} |b_{ij}|^{2} = \|B\|_{F}^{2}$$

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UNITARY MATRICES AND PLANE ROTATIONS : 2-D CASE

- Consider rotating the 2-D Euclidean plane counter-clockwise by an angle θ .
- Resulting coordinates,

$$x' = x \cos \theta - y \sin \theta$$
$$y' = x \sin \theta + y \cos \theta$$

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• Equivalently,

$$\begin{bmatrix} x'\\y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta\\\sin\theta & \cos\theta \end{bmatrix}$$
• Note that $U = \begin{bmatrix} \cos\theta & -\sin\theta\\\sin\theta & \cos\theta \end{bmatrix}$ is unitary.

UNITARY MATRICES AND PLANE ROTATIONS : GENERAL

CASE

$$U(\theta, 2, 4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$



- $U(\theta,2,4)$ rotates the second and fourth axes in ${f R}^4$ counter clock-wise by $\theta.$
- The other axes are not changed.
- Left multiplication by $U(\theta,2,4)$ affects only rows 2 and 4.
- Note that $U(\theta, 2, 4)$ is unitary.
- Such $U(\theta, m, n)$ are called Givens rotations.

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PRODUCT OF GIVENS ROTATIONS

- $U = U(\theta_1, 1, 3)U(\theta_2, 2, 4)$ rotates
 - second and fourth axes in \mathbf{R}^4 counter clock-wise by θ_2 .
- first and third axes in \mathbf{R}^4 counter clock-wise by θ_1 .
- U is unitary \Rightarrow product of Givens rotations is unitary.
- Such matrices are used in Least-Squares and eigenvalue computations.

SPECIAL UNITARY MATRICES: HOUSEHOLDER MATRICES

Let $w \in \mathbf{C}^n$ be a normalized ($w^*w = 1$) vector and define

 $U_w = I - 2ww^*$

Properties:



1. U_w is unitary and Hermitian.

2. $U_w x = x, \forall x \perp w.$

- 3. $U_w w = -w$
- 4. There is a Householder matrix such that

 $y = U_w x$

for any given *real* vectors x and y of the same length.

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QR-FACTORIZATION

Thm: If $A \in M_{n,m}$ and n > m, then

A = QR

with $Q \in M_{n,m}$ such that $Q^*Q = I$ and $R \in M_m$ is upper triangular.

- If m = n then Q is unitary.
- If A is nonsingular, then the diagonal elements of R can be taken to be positive (Q and R are in this case unique).
- Can be described as Gram Schmidt orthogonalization combined with book keeping.
- Alternative algorithm: Series of Householder transformations.
- Useful in Least squares solutions, eigenvalue computations etc.

SCHUR'S UNITARY TRIANGULARIZATION THM

Theorem: Given $A \in M_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, there is a unitary matrix $U \in M_n$ such that

$$U^*AU = T = [t_{ij}]$$

is upper triangular with $t_{ii} = \lambda_i$ (i = 1, ..., n) in any prescribed order. If $A \in M_n(\mathbf{R})$ and all λ_i are real, U may be chosen real and orthogonal.



Consequence: Any matrix in M_n is unitarily similar to an upper (or lower) triangular matrix. Note that $A = UTU^*$.

Uniqueness:

- 1. Neither U nor T is unique.
- 2. Eigenvalues can appear in any order

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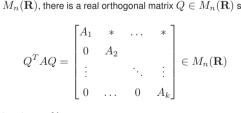
3. Two triangular matrices can be unitarily similar

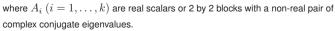
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SCHUB: THE GENERAL REAL CASE

Given $A \in M_n(\mathbf{R})$, there is a real orthogonal matrix $Q \in M_n(\mathbf{R})$ such that





IMPLICATIONS OF THE SCHUR THEOREM

1.
$$\operatorname{tr} A = \sum_j \lambda_j(A)$$

2. $\det A = \prod_j \lambda_j(A)$

4. ...

3. Cayley-Hamilton theorem.



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CAYLEY-HAMILTON THEOREM

Let $p_A(t) = \det(tI - A)$ be the characteristic polynomial of $A \in M_n$. Then

 $p_A(A) = 0$

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Consequences:

- $A^{n+k} = q_k(A) \ (k \ge 0)$ for some polynomials $q_k(t)$ of degrees $\le n 1$.
- If A is nonsingular: $A^{-1}=q(A)$ for some polynomial q(t) of degree $\leq n-1.$

Important : Note $p_A(C)$ is a matrix valued function.

NORMAL MATRICES

Def: A matrix $A \in M_n$ is *normal* if $A^*A = AA^*$.

Examples: All unitary matrices are normal. All Hermitian matrices are normal.



Def: $A \in M_n$ is unitarily diagonalizable if A is unitarily equivalent to a diagonal matrix.

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FACTS FOR NORMAL MATRICES

The following are equivalent:

- 1. A is normal
- 2. A is unitarily diagonalizable



3. $\|A\|_F^2 \triangleq \sum_{i,j} |a_{ij}|^2 = \sum_{i=1}^n |\lambda_i|^2$

4. there is an orthonormal set of \boldsymbol{n} eigenvectors of \boldsymbol{A}

The equivalence of 1 and 2 is called "the Spectral Theorem for Normal matrices."

IMPORTANT SPECIAL CASE: HERMITIAN (SYM) MATRICES **CANONICAL FORMS** Spectral theorem for Hermitian matrices: An equivalence relation partitions the domain. If $A \in M_n$ is Hermitian, then, • Simple to study equivalence if two objects in an equivalence class can be • all eigenvalues are real related to one representative object. • A is unitarily diagonalizable. Requirements of the representatives - Belong to the equivalence class. If $A \in M_n(\mathbf{R})$ is symmetric, then A is real orthogonally diagonalizable. КТН - One per class. • Set of such representatives is a Canonical form • We are interested in a canonical form for equivalence relation defined by similarity. KTH - Signal Processing Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson KTH - Signal Processing Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson 17 19 SVD: SINGULAR VALUE DECOMPOSITION **CANONICAL FORMS: JORDAN FORM Theorem:** Any $A \in M_{m,n}$ can be decomposed as Every equivalence class under similarity contains essentially only one, so called. Jordan matrix: $J = \begin{bmatrix} J_{n_1}(\lambda_1) & 0 \\ & \ddots & \\ & & \end{bmatrix}$ $A = V\Sigma W^*$ • $V \in M_m$: Unitary with columns being eigenvectors of AA^* . 0 $J_{n_k}(\lambda_k)$ • $W \in M_n$: Unitary with columns being eigenvectors of A^*A . KTH where each block $J_k(\lambda) \in M_k$ has the structure • $\Sigma = [\sigma_{ii}] \in M_{m,n}$ has $\sigma_{ii} = 0, \forall i \neq j$ $J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & & \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & \lambda & 1 \\ 0 & & & \ddots & \ddots \end{bmatrix}$ Suppose $\operatorname{rank}(A) = k$ and $q = \min\{m, n\}$, then • $\sigma_{11} \geq \cdots \geq \sigma_{kk} > \sigma_{k+1,k+1} = \cdots = \sigma_{aa} = 0$ • $\sigma_{ii} \equiv \sigma_i$ square roots of non-zero eigenvalues of AA^* (or A^*A) • Unique : σ_i , Non-unique : V, WKTH - Signal Processing Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtssor KTH - Signal Processing Magnus Jansson / Bhavani Shankar / Joakim Jaldén / Mats Bengtsson 18 20

THE JORDAN FORM THEOREM

Note that the orders n_i or λ_i are generally not distinct.

Theorem: For a given matrix $A \in M_n$, there is a nonsingular matrix $S \in M_n$ such that $A = SJS^{-1}$ and $\sum_i n_i = n$. The Jordan matrix is unique up to permutations of the Jordan blocks.

The Jordan form may be numerically unstable to compute but it is of theoretical interest.

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JORDAN FORM CONT'D

- The number k of Jordan blocks is the number of linearly independent eigenvectors. (Each block is associated with an eigenvector from the standard basis.)
- J is diagonalizable iff k = n.

КТН

- The number of blocks corresponding to the same eigenvalue is the geometric multiplicity of that eigenvalue.
- The sum of the orders (dimensions) of all blocks corresponding to the same eigenvalue equals the algebraic multiplicity of that eigenvalue.

APPLICATIONS OF THE JORDAN FORM

Linear systems:

$$\dot{x}(t) = Ax(t); \ x(0) = x_0$$

The solution may be "easily" obtained by changing state variables to the Jordan form.



Convergent matrices: If all elements of A^m tend to zero as $m \to \infty$, then A is a convergent matrix. Fact: A is convergent iff $\rho(A) < 1$ (that is, iff $|\lambda_i| < 1, \ \forall i$). This may be proved, e.g., by using the Jordan canonical form.

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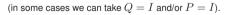
Linear systems of equations are easy to solve if we can factorize the system matrix as A = LU where L(U) is lower (upper) triangular.

Theorem: If $A \in M_n$, then there exist permutation matrices $P, Q \in M_n$ such that

TRIANGULAR FACTORIZATIONS



A = PLUQ



WHEN TO USE WHAT?

	Theoretical	Practical
	derivations	implem.
Schur triangularization	\odot	(:)
QR factorization	\odot	(;)
Spectral dec.	\odot	(?)
SVD	\odot	:
Jordan form	\odot	::

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