Homework #4

Numbers below refer to problems in Horn, Johnson "Matrix analysis." A number 1.1.P.2 refers to Problem 2 in Section 1.1.

1. (4.1.P19) Let $A \in M_n$ be a projection ($A^2 = A$). One says that A is a *Hermitian projection* if A is Hermitian and that A is an *orthogonal projection* if the range of A is orthogonal to its null space. Use the basic properties of Hermitian matrices to show that A is a Hermitian projection if and only if it is an orthogonal projection.

   Hint: $x = (I-A)x+Ax$ is a sum of vectors in the null space and range of A. If the null space is orthogonal to the range, then $x^*Ax = ((I-A)x+Ax))^*Ax = x^(A^*A)x$ is real for all x.

2. Prove that the formulation of Courant-Fischer's max-min theorem shown in the lecture slides (Theorem 4.2.6 in the 2nd edition of the book) is equivalent to

$$\lambda_k = \min_{w_1,\ldots,w_{n-k}} \max_{\substack{x\neq 0 \\ x\perp w_1,\ldots,w_{n-k}}} \frac{x^*Ax}{x^*x}$$

$$\lambda_k = \max_{w_1,\ldots,w_{k-1}} \min_{\substack{x\neq 0 \\ x\perp w_1,\ldots,w_{k-1}}} \frac{x^*Ax}{x^*x}$$

   where $w_i, x \in \mathbf{C}^n$ and the vectors $\{w_i\}$ are allowed to be linearly dependent.

3. (4.3.P4, 4.3.P14 in the old edition) If $A, B \in M_n$ are Hermitian and their eigenvalues are arranged in nondecreasing order, explain why $\lambda_i(A+B) \leq \min\{\lambda_j(A) + \lambda_k(B) : j+k = i+n\}$.

4. (4.4.P2) Provide details for the following derivation of the Autonne-Takagi factorization, using real valued representations. Let $A \in M_n$ be symmetric. If A is singular and rank $A = r$, it is unitarily congruent to $A' \oplus 0_{n-r}$, in which $A' \in M_r$ is non-singular and symmetric (no need to prove this step). Assume therefore WLOG that A is nonsingular. Let $A = A_1 + iA_2$ with A_1, A_2 real and let $x, y \in \mathbb{R}^n$. Consider the real representation $R_2(A) = \begin{bmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{bmatrix}$, in which A_1, A_2 and $R_2(A)$ are symmetric. Show that

   (a) $R_2(A)$ is nonsingular.

   (b) $R_2(A)\begin{bmatrix} x \\ -y \end{bmatrix} = \lambda \begin{bmatrix} x \\ -y \end{bmatrix}$ if and only if $R_2(A)\begin{bmatrix} x \\ -y \end{bmatrix} = -\lambda \begin{bmatrix} x \\ -y \end{bmatrix}$, so the eigenvalues of $R_2(A)$ appear in $\pm$ pairs.

(c) Let $\begin{bmatrix} x_1 \\ -y_1 \end{bmatrix}, \ldots, \begin{bmatrix} x_n \\ -y_n \end{bmatrix}$ be orthonormal eigenvectors of $R_2(A)$ associated with its positive eigenvalues $\lambda_1, \ldots, \lambda_n$, let $X = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 & \ldots & y_n \end{bmatrix}$, $\Sigma = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $V = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$ and $\Lambda = \Sigma \oplus (-\Sigma)$. Then V is real orthogonal and $R_2(A) = V\Lambda V^T$. Let $U = X - iY$. Explain why U is unitary and show that $U\Sigma U^T = A$.

5. a) Let $\alpha = [\alpha_i] \in \mathbf{R}^n$ and $\beta = [\beta_i]$, where $\beta_1 = \cdots = \beta_n = \frac{1}{n}\sum \alpha_i$. Show that α majorizes β.

   b) [Optional, only the solution to a) is considered in the grading] Let $\Lambda = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$. Try to find a unitary matrix $U \in M_n$ such that all diagonal elements of $U\Lambda U^*$ are equal. Note that this is a simple special case of Theorem 4.3.48. However, in this special case, it is easy to determine a matrix U that works for all α (in general, U will have to depend on the two vectors).

6. (4.5.P8) Let $A, S \in M_n$ with A Hermitian and S nonsingular. Let the eigenvalues of A and SAS^* be arranged in nondecreasing order. Let $\lambda_k(A)$ be a nonzero eigenvalue. Deduce the relative eigenvalue perturbation bound $|\lambda_k(SAS^*) - \lambda_k(A)|/|\lambda_k(A)| \leq \rho(I - SS^*)$ from Ostrowski's theorem. What does this say if S is unitary? If S is "close to unitary"?

7. Consider the quadratic form $Q(x, y, z) = x^2 + 4xy + 2xz + 2y^2 + 8yz - z^2$ in the real valued variables x, y, z. One way to complete the squares is $Q(x, y, z) = (x+2y+z)^2 - 2(y-z)^2$. Show that this expression can be written as $Q(x, y, z) = v^T S^T \operatorname{diag}(1, -2, 0) S v$ where $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ * & * & * \end{bmatrix}$ (the last row is arbitrary).

   Determine at least one alternative way to complete the squares. Use Sylvester's law of inertia to show that you will always get exactly one positive squared term and one negative squared term (and no non-zero third term), no matter how you complete the squares.

8. Let $A = A^* \in M_n$ be a positive definite matrix ($\lambda_i(A) > 0$). Show that

$$\log\det(A) - \operatorname{Tr}(A)$$

   is maximized by $A = I$.