LECTURE 4: OUTLINE

- Chapter 4: Hermitian and symmetric matrices, Congruence

LECTURE 4: HERMITIAN MATRICES

Def: A matrix $A = [a_{ij}] \in M_n$ is *Hermitian* if $A = A^*$.
A is *skew-Hermitian* if $A = -A^*$.

Simple observations:

1. If A is Hermitian, then A^k and A^{-1} are Hermitian.
2. $A + A^*$ and AA^* are Hermitian and $A - A^*$ is skew-Hermitian for all $A \in M_n$.
3. Any $A \in M_n$ can be decomposed uniquely as $A = B + iC = B + D$ where B, C are Hermitian and D skew-Hermitian. In fact

$$B = \frac{1}{2}(A + A^*) \quad D = iC = \frac{1}{2}(A - A^*)$$

4. A Hermitian matrix in M_n is completely described by n^2 real valued parameters.

HERMITIAN MATRICES CONT'D

A is Hermitian iff

- x^*Ax is real for all $x \in \mathbf{C}^n$
- A is normal with real eigenvalues
- S^*AS is Hermitian for all $S \in M_n$

All eigenvalues of a Hermitian matrix are real and it has a complete set of orthonormal eigenvectors (the last fact follows as a special case of the spectral theorem for normal matrices).

Thm (spectral): $A \in M_n$ is Hermitian iff it is unitarily diagonalizable to a real diagonal matrix. A matrix A is real symmetric iff it can be diagonalized by a real orthogonal matrix to a real diagonal matrix.

COMMUTATION OF HERMITIAN MATRICES

Let $\mathcal{F}$ be a family of Hermitian matrices. Then all $A \in \mathcal{F}$ are simultaneously unitarily diagonalizable iff $AB = BA$ for all $A, B \in \mathcal{F}$.

Positive Definiteness

A Hermitian matrix $A \in M_n$ is

Positive definite if $x^*Ax > 0$ for all $x \in \mathbb{C}^n$, $x \neq 0$.

Positive semidefinite if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$, $x \neq 0$.

Negative definite if $x^*Ax < 0$ for all $x \in \mathbb{C}^n$, $x \neq 0$.

Negative semidefinite if $x^*Ax \leq 0$ for all $x \in \mathbb{C}^n$, $x \neq 0$.

Indefinite if there are $y, z \in \mathbb{C}^n$ with $y^*Ay < 0 < z^*Az$.

Much more in positive (semi)definiteness in Chapter 7

Variational characterization of eigenvalues

Let $A \in M_n$ be Hermitian with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$.
Thm (Rayleigh-Ritz):

$$\lambda_1 = \min_{x \neq 0} \frac{x^*Ax}{x^*x} = \min_{x^*x=1} x^*Ax$$
$$\lambda_n = \max_{x \neq 0} \frac{x^*Ax}{x^*x} = \max_{x^*x=1} x^*Ax$$

Thm (Courant-Fischer): Let S denote a subspace of $\mathbf{C}^n$. Then,

$$\lambda_k = \min_{\{S:\dim[S]=k\}} \max_{\substack{x \in S \\ x \neq 0}} \frac{x^*Ax}{x^*x}$$

$$\lambda_k = \max_{\{S:\dim[S]=n-k+1\}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^*Ax}{x^*x}$$

Applications of C-F thm

Thm: If $A, B \in M_n$ are Hermitian, then if $j + k \geq n + 1$

$$\lambda_{j+k-n}(A + B) \leq \lambda_j(A) + \lambda_k(B)$$

and if $j + k \leq n + 1$

$$\lambda_j(A) + \lambda_k(B) \leq \lambda_{j+k-1}(A + B)$$

Applications cont'd

Thm: If $A, B \in M_n$ are Hermitian, then

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B)$$

Interlacing theorem: Let $z \in \mathbf{C}^n$ and $A \in M_n$ be Hermitian. Then, for $k = 1, 2, \ldots, n - 1$:

$$\lambda_k(A + zz^*) \leq \lambda_{k+1}(A) \leq \lambda_{k+1}(A + zz^*)$$
$$\lambda_k(A) \leq \lambda_k(A + zz^*) \leq \lambda_{k+1}(A)$$

$$\lambda_k(A - zz^*) \leq \lambda_k(A) \leq \lambda_{k+1}(A - zz^*)$$
$$\lambda_k(A) \leq \lambda_{k+1}(A - zz^*) \leq \lambda_{k+1}(A)$$

Applications cont'd

Interlacing theorem for bordered matrices:
Let $A \in M_n$ be Hermitian, $y \in \mathbf{C}^n$, $a \in \mathbf{R}$ and define

$$\hat{A} = \begin{bmatrix} A & y \\ y^* & a \end{bmatrix}$$

Then with $\lambda_i \in \sigma(A)$ and $\hat{\lambda}_i \in \sigma(\hat{A})$

$$\hat{\lambda}_1 \leq \lambda_1 \leq \hat{\lambda}_2 \leq \cdots \leq \hat{\lambda}_n \leq \lambda_n \leq \hat{\lambda}_{n+1}$$

The Poincaré separation theorem

Let $A \in M_n$ be Hermitian, let $U \in M_{n,r}$ be a matrix with $r \leq n$ orthonormal columns and define $B_r = U^* A U$. Then

$$\lambda_k(A) \leq \lambda_k(B_r) \leq \lambda_{k+n-r}(A); \qquad k = 1, 2, \ldots, r$$

Application:

$$\min_{U,\ U^*U=I_r} \mathrm{Tr}(U^* A U) = \sum_{k=1}^{r} \lambda_k(A)$$

$$\max_{U,\ U^*U=I_r} \mathrm{Tr}(U^* A U) = \sum_{k=1}^{r} \lambda_{k+n-r}(A)$$

Note that equality is obtained by choosing the columns of U as suitable eigenvectors of A.

Generalized Rayleigh Quotients

Let $A \in M_n$ be Hermitian and $B \in M_n$ be Hermitian positive definite. Consider the following **generalized eigenvalue problem**

$$Ax = \lambda Bx$$

with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then,

$$\lambda_1 = \min_{x \neq 0} \frac{x^* A x}{x^* B x} = \min_{x^* B x \geq 1} x^* A x$$

$$\lambda_n = \max_{x \neq 0} \frac{x^* A x}{x^* B x} = \max_{x^* B x \leq 1} x^* A x$$

Solve the generalized eigenvalue problem in Matlab using
`[E,Lambda]=eig(A,B);`

Note: Elements of `Lambda` not sorted.

Majorization

Def: Let $\alpha = [\alpha_i] \in \mathbf{R}^n$ and $\beta = [\beta_i] \in \mathbf{R}^n$ with sorted versions,
$\alpha_{j_1} \leq \alpha_{j_2} \leq \cdots \leq \alpha_{j_n}$ and $\beta_{m_1} \leq \beta_{m_2} \leq \cdots \leq \beta_{m_n}$.

If

$$\sum_1^n \alpha_i = \sum_1^n \beta_i$$

and

$$\sum_{i=1}^{k} \beta_{m_i} \leq \sum_{i=1}^{k} \alpha_{j_i}$$

for all $k = 1, 2, \ldots, n$, then the vector β *majorizes* the vector α.

Note: The notation is not standardized, some texts (including 1st edition of Horn&Johnson) use the opposite definition.

Majorization cont'd

Thm: Let $A \in M_n$ be Hermitian. The vector of eigenvalues majorizes the vector of diagonal elements.

Converse thm: If the vector $\lambda \in \mathbf{R}^n$ majorizes the vector $a \in \mathbf{R}^n$ then there exists a real symmetric matrix $A \in M_n(\mathbf{R})$ with a_i as diagonal elements and λ_i as eigenvalues.

Thm: Let $A, B \in M_n$ be Hermitian and let $\lambda(A)$ be the sorted vector of eigenvalues of A etc. The vector $\lambda(A) + \lambda(B)$ majorizes the vector $\lambda(A+B)$.

Complex symmetric matrices

Autonne-Takagi factorization: If $A \in M_n$ is symmetric, then $A = U\Sigma U^T$. Here, $U \in M_n$ and unitary, $\Sigma = diag\{\sigma_1, \dots, \sigma_n\}$ is real and nonnegative. The columns of U can be taken as an orthonormal set of eigenvectors to $A\bar{A}$ and σ_i is the square root of an eigenvalue of $A\bar{A}$.

Thm: Every matrix $A \in M_n$ is similar to a symmetric matrix.

Thm: Let $A \in M_n$. There exist a nonsingular matrix S and a unitary matrix U such that $(US)A(\bar{U}S)^{-1}$ is a diagonal matrix with nonnegative elements.

Congruence

Def: Let $A, B \in M_n$ and S a nonsingular matrix.
If $B = SAS^*$, then B is *-congruent to A.
If $B = SAS^T$, then B is ^{T}-congruent to A.

Both congruence relations induce equivalence classes:

1. A is congruent to A
2. If A is congruent to B, then B is congruent to A.
3. If A is congruent to B and B is congruent to C, then A is congruent to C.

Inertia

Def: Let $A \in M_n$ be Hermitian. The *inertia* of A is the ordered triple

$$i(A) = (i_+(A), i_-(A), i_0(A))$$

where the entries correspond to the number of positive, negative and zero eigenvalues of A, respectively.
Note that the rank of A equals $i_+(A) + i_-(A)$.
The *signature* of A is $i_+(A) - i_-(A)$.

CANONICAL FORM/SYLVESTER'S LAW OF INERTIA

If $A \in M_n$ is Hermitian, then we can decompose it as

$$A = SI(A)S^*$$

where S is nonsingular and $I(A)$ is the *inertia matrix*

$$I(A) = \mathrm{diag}(1 \; \ldots \; 1 \; -1 \; \ldots \; -1 \, 0 \ldots 0)$$

Thm (Syl): Let $A, B \in M_n$ be Hermitian. Then $A = SBS^*$ for a nonsingular matrix $S \in M_n$ iff A and B have the same inertia.

QUANTITATIVE INERTIA RESULT / ^{T}-CONGRUENCE

Thm: (Ostrowski) Let $A, S \in M_n$ where A is Hermitian. Let the eigenvalues be arranged in nondecreasing order. For each $k = 1, \ldots, n$ there exists a real number θ_k such that $\lambda_1(SS^*) \leq \theta_k \leq \lambda_n(SS^*)$ and

$$\lambda_k(SAS^*) = \theta_k \lambda_k(A)$$

Thm: Let $A, B \in M_n$ be symmetric matrices (real or complex). There is a nonsingular matrix $S \in M_n$ such that $A = SBS^T$ iff A and B have the same rank.

More about diagonalization by congruence: Thm 4.5.17 (4.5.15 in old ed.)