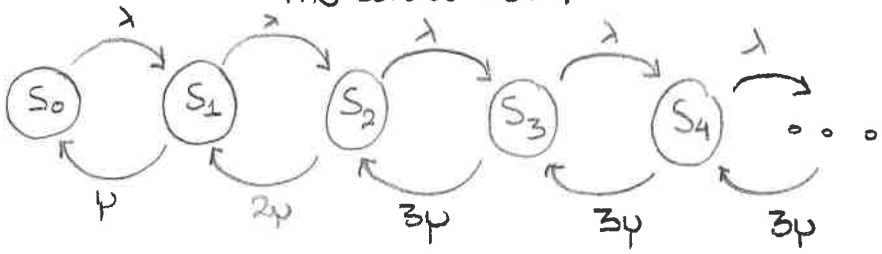
Exam 2014, March.

1. $M/M/3$, $\lambda = 20h^{-1}$, $\mu = 12h^{-1}$, $\rho = \frac{\lambda}{\mu} = \frac{20}{12} = \frac{5}{3}$, $m = 3$

State space: S_k: k customers in the service desk



b) Server Utilization: $U = \frac{\rho}{m} = \frac{5/3}{3} = \frac{5}{9}$

c) i] $P_{WAIT} = D_m(\rho) = \frac{m E_m(\rho)}{m - \rho(1 - E_m(\rho))}$ [formula sheet]

$E_m(\rho) = E_3\left(\frac{5}{3}\right) \cong 0{,}1573$ [Erlang Tables]

$\Rightarrow \boxed{P_{WAIT} \cong 0{,}297}$

ii] P(finds the desk idle) $= P_0$ (homogeneous)

Balance Equations

$\lambda P_0 = \mu P_1 \longrightarrow P_1 = \rho P_0$ (1)

$\lambda P_1 = 2\mu P_2 \longrightarrow P_2 = \frac{\rho^2}{2} P_0$ (2)

$P_0 + P_1 + P_2 = 1 - P_{WAIT}$ (3)

$\Rightarrow \boxed{P_0 = 0{,}2076}$

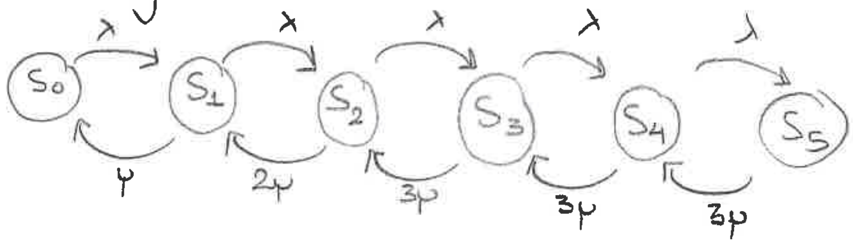
$P_1 = 0{,}346$

$P_2 = 0{,}2883$

iii] $P(W > 2min) = 1 - F_W(2min) = D_m(\rho) \cdot e^{-\mu(m-\rho)t} =$ | $t = 1/30 h$

$= 0{,}297 \cdot e^{-0{,}533} \cong \boxed{0{,}1742}$

d) State Diagram



$\lambda_{eff} = \lambda \cdot (1 - P_5)$

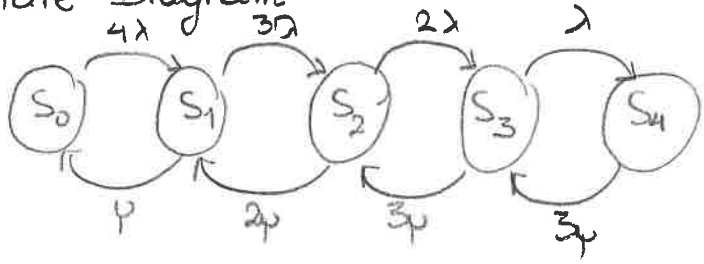
2.

4 students

$m = 3$ servers

$\lambda = 1\,h^{-1}$, $\mu = 3h^{-1}$

State Diagram



a. M/M/3/4/4

b. Balance Equations

$$\left.\begin{aligned} 4\lambda P_0 &= \mu P_1 \\ 3\lambda P_1 &= 2\mu P_2 \\ 2\lambda P_2 &= 3\mu P_3 \\ \lambda P_3 &= 3\mu P_4 \\ \sum_{k=0}^{4} P_k &= 1 \end{aligned}\right\} \Longrightarrow \begin{cases} P_0 = 0{,}316 \\ P_1 = 0{,}4213 \\ P_2 = 0{,}2107 \\ P_3 = 0{,}0468 \\ P_4 = 0{,}0052 \end{cases}$$

$$\bar{N} = \sum k P_k = P_1 + 2P_2 + 3P_3 + 4P_4 = 1{,}0039$$

No losses: $\lambda eff = \bar{\lambda} = 4\lambda P_0 + 3\lambda P_1 + 2\lambda P_2 + \lambda P_3 = 2{,}9961$ / hour

In 4 hours: $2{,}9961 \cdot 4 = \boxed{11{,}9844 \text{ tasks}}$

c. Utilization: $U = \dfrac{\lambda eff \cdot E[T_{serv}]}{m} = \dfrac{2{,}9961/3}{3} \simeq \boxed{1/3}$

Little: $\bar{T}_{SYST} = \dfrac{\bar{N}}{\lambda eff} = \dfrac{1{,}0039}{2{,}9961} \simeq \boxed{0{,}3351} \simeq \boxed{1/3 \text{ hour}}$

d. P(no servers idle) $= P_3 + P_4 = \boxed{0{,}052}$

e. P(can't start immeduately) $= P$(arrival sees S_3) $= \dfrac{\lambda P_3}{\lambda eff} = 0{,}0156$

f. [Server sharing] $\longrightarrow$ State diagram does not change

P(an arriving task can't run at full speed) $= P$(sees S_3) $= \dfrac{\lambda_3 P_3}{\lambda eff} = 0{,}0156$

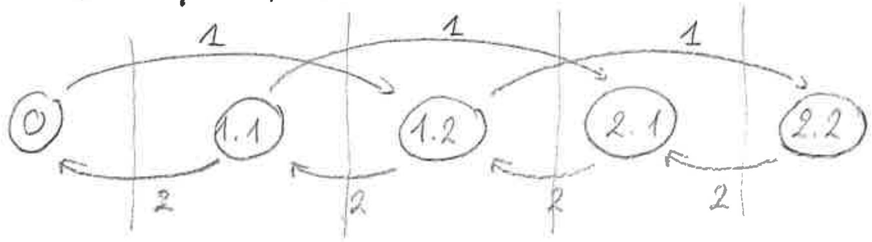
(3)

$X = X_1 + X_2 \Rightarrow X \sim$ Erlang-2

$\mu_1 = \mu_2 = 2$

$\lambda = 1$

a) $M / E_2 / 1 / 2$



$p_0 = 2 p_{11}$

$p_0 + p_{11} = 2 p_{12}$

$p_{11} + p_{12} = 2 p_{21}$

$p_{12} = 2 p_{22}$

$p_{11} = \frac{p_0}{2}$

$p_{12} = \frac{p_0 + \frac{p_0}{2}}{2} = \frac{3}{4} p_0$

$p_{21} = \frac{\frac{p_0}{2} + \frac{3}{4} p_0}{2} = \frac{5}{8} p_0$

$p_{22} = \frac{3}{8} p_0$

$p_0 \left(1 + \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{3}{8}\right) = 1$

$p_0 \frac{8 + 4 + 6 + 5 + 3}{8} = 1$

$p_0 = \frac{8}{26} = \frac{4}{13}$

$p_{11} = \frac{2}{13}$, $p_{12} = \frac{3}{13}$, $p_{21} = \frac{5}{26}$, $p_{22} = \frac{3}{26}$

b) $P(\text{empty}) = p_0 = \frac{4}{13}$

$P(\text{block}) = p_{21} + p_{22} = \frac{8}{26} = \frac{4}{13}$

c) $N_S = 1 - p_0 = \frac{9}{13}$, $N = (p_{11} + p_{12}) \cdot 1 + (p_{21} + p_{22}) 2 = \frac{5}{13} + \frac{8}{13} = 1$, $N_q = \frac{4}{13}$

$T = \frac{N}{\lambda_{ef}} = 1 \cdot \frac{1}{1 - P(\text{block})} = \frac{13}{9}$

d) $T = P(\text{arrives to state } 1.1) \cdot \frac{1}{\mu} + P(\text{arrives to state } 1.2) \cdot \frac{2}{\mu} =$

$$\frac{\frac{2}{13} \cdot \frac{1}{2} + \frac{3}{13}}{\frac{5}{13}} = \frac{4}{5}$$

(4)

Type 1 $\quad X_1 = 1 \qquad\qquad \lambda_1 = 0.3 \cdot \frac{1}{10}$

Type 2 $\quad X_2 \sim Exp(\mu) \quad E[X_2] = 5, \quad \mu = \frac{1}{5} \qquad \lambda_2 = 0.7 \cdot \frac{1}{10}$

$E_2[X_2] = 50$

a) $E[X] = p_1 \cdot E[X_1] + p_2 \, E[X_2] = 0.3 \cdot 1 + 0.7 \cdot 5 = 3.8$

$E[X^2] = p_1 \, E[X_1^2] + p_2 \, E[X_2^2] = 0.3 \cdot 1 + 0.7 \cdot 2 \cdot 25 = 35.3$

$V[X] = E[X^2] - E[X]^2 = 20.86$

$C_X^2 = \frac{V[X]}{E[X]^2} = 1.444 \quad > 1$, as expected.

b) M / G / 1 $\qquad \rho = \lambda E[X] = 0.38$

$P(\text{idle}) = 1 - \rho = 1 - \lambda E[X] = 0.62$

$\bar{W} = \frac{\lambda E[X^2]}{2(1-\rho)} = \underline{\underline{2.84}}$

$\bar{T} = \bar{W} + E[X] = 6.648$

Note, the system can not be represented with a Markov Chain, due to the fixed Type 1 service time!

c) Type 1 with preemptive priority

$\bar{W}_1 = \bar{W}_{M/D/1} = \frac{\lambda_1 E[X_1^2]}{2(1-\rho_1)} = \underline{\underline{0.015}}$

$\lambda_1 = 0.3 \cdot \frac{1}{10}$

$\rho_1 = \frac{0.3}{10} = 0.03$

$\rho_2 = \frac{0.7}{10} \cdot 5 = 0.35$

$\bar{W}_2 = \frac{R_{S,2}}{(1-\rho_1)(1-\rho_1-\rho_2)} = \frac{17.6}{0.97 \cdot 0.62} = \underline{\underline{2.92}}$

$R_{S,2} = \frac{1}{2}\left(\lambda_1 E[X_1^2] + \lambda_2 E[X_2^2]\right) =$

$= \frac{1}{2} \cdot \lambda E[X^2] = \frac{35.3}{20} = 1.76$

W_1 decreased significantly. W_2 increased a little (as ρ_1 was low...)

d) The service of this customer starts immediately, but is interrupted by Type 1 customers:

$E[X_2'] = E[X_2] + E[X_2'] \cdot \lambda_1 \cdot E[X_1]$

$E[X_2'] = \frac{E[X_2]}{1 - \rho_1} = \frac{5}{0.97} = 5.15 \text{ s}$

(5)

a) $C = 240 \cdot 10^3$ bit/s

$\lambda = 100$ packets/s

$E[L] = 150$ Byte $= 1.2 \cdot 10^3$ bits

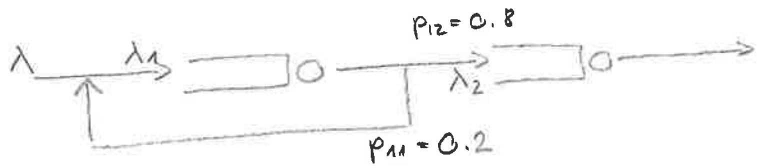
$E[x] = \frac{E[L]}{C} = \underset{5 \cdot 10^{-3} s}{0.005 s}$, $E[x^2] = 50 \cdot 10^{-6}$

$\rho = \lambda E[x] = 0.5$

$V = 0.1 s$ (fixed) $\Rightarrow E[V] = 0.1$, $E[V^2] = E[V]^2$

$$\underline{\underline{W}} = \frac{\lambda E[x^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]} = \frac{5 \cdot 10^{-3}}{1} + \frac{0.1}{2} = \underline{\underline{0.055 s}}$$

b) $\mu_1 = 3$, $\mu_2 = 2$, $\lambda = 1.6$ [jobs/sec]



$\lambda_1 = \lambda + 0.2 \lambda_1$

$\lambda_1 = \frac{\lambda}{0.8} = 2$ $\qquad \rho_1 = \frac{\lambda_1}{\mu_1} = \frac{2}{3}$

$\lambda_2 = \lambda_1 \cdot 0.8 = 1.6$ $\qquad \rho_2 = \frac{\lambda_2}{\mu_2} = \frac{8}{10}$

$P(\text{nw empty}) = P(\text{queue 1 empty}) P(\text{queue 2 empty}) = \frac{1}{3} \cdot \frac{2}{10} = \frac{1}{15}$

$N = N_1 + N_2 = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} = 2 + 4 = 6$

$T = \frac{N}{\lambda} = \frac{6}{1.6} = 3.75 s$

5c.

M/M/10/10

$\lambda = 10$, $\mu = 1$ $\longrightarrow \rho = 10$

$P_{block} = E_{10}(10) = 0{,}2145$

$\bar{T}_B = \frac{1}{10\mu} = 0{,}1$, $\frac{\bar{T}_B}{\bar{T}_B + \bar{T}_{NB}} = 0{,}2145 \longrightarrow \bar{T}_{NB} = 0{,}3661$

5d.

M/M/1

$\lambda = 5$
$\mu = 8$ $\longrightarrow \rho = 5/8$

State diagram $\rightarrow$ standard M/M/1 $\longrightarrow \bar{N} = \frac{\rho}{1-\rho} = \frac{5/8}{3/8} = 5/3$

$\bar{T}_{SYST} = \bar{N}/\lambda = \frac{5/3}{5} = \frac{1}{3}$ (Little)

$\bar{W} = \bar{T}_{SYST} - \bar{T}_{SERV} = \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$

* Can also be solved with M/M/1 waiting formula