MATRIX ALGEBRA
MAGNUS JANSSON/MATS BENGTSSON
Deadline: 2014–05–02

Homework # 5

Numbers below refer to problems in Horn, Johnson "Matrix analysis." A number 1.1.P2 means Problem 2 in Section 1.1.

1. (part of 5.1.P4) Let $\|\cdot\|$ be a norm on V that is derived from an inner product. Show that it satisfies the *parallelogram identity*

$$\frac{1}{2}(\|x+y\|^2 + \|x-y\|^2) = \|x\|^2 + \|y\|^2$$

   for all $x, y \in V$. Why is this identity so named?

   It can be shown that the parallelogram identity is necessary and sufficient for a given norm to be derived from an inner product; see (5.1.P12).

2. (5.1.P9) Let $\|\cdot\|$ be a norm on V that is derived from an inner product, let $x, y \in V$ and suppose that $y \neq 0$. Show that

   (a) the scalar α_0 that minimizes the value of $\|x - \alpha y\|$ is $\alpha_0 = \frac{<x,y>}{\|y\|^2}$, and

   (b) $x - \alpha y$ and y are orthogonal.

   Hints: Make sure that your proof holds for all inner products that satisfy the axiomatic definition. Note that the inner product in general is not necessarily differentiable.

3. (5.2.P1) If $0 < p < 1$, then $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ defines a function on $\mathbb{C}^n$ that satisfies all but one of the axioms for a norm. Which one fails? Give an example.

4. (part of 5.4.P3) Verify the entries in the following table of bounds of the form $\|x\|_\alpha \leq C_{\alpha\beta}\|x\|_\beta$.

$[C_{\alpha\beta}] =$

$\alpha \backslash \beta$	1	2	∞
1	1	$\sqrt{n}$	n
2	1	1	$\sqrt{n}$
∞	1	1	1

   For each entry, show that the corresponding bound is the best possible, by providing a nonzero vector x for which the bound is attained with equality.

5. (5.4.P5) Show that the functions f_k in Example 5.4.2 have the property that

   (a) $f_k(x) \to 0$ as $k \to \infty$ for each given x,

   (b) $\|f_k - f_j\|_1 \to 0$ as $k, j \to \infty$, and

   (c) for each $k \geq 2$, there is some $J > k$ for which $\|f_k - f_j\|_\infty > k^{1/2}$ for all $j > J$.

   Thus, a sequence in an infinite dimensional normed linear space can be convergent in one sense (pointwise), be Cauchy in one norm, and not be Cauchy in another norm.

6. (5.4.P8, missing in old edition) Show that the dual norm of the $k-$norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ is

$$\|x\|_{[k]}^D = \max\left\{\frac{1}{k}\|y\|_1, \|x\|_\infty\right\}$$

   Hint: The $k-$norm is defined as the sum of the k largest magnitudes of the entries in x, i.e.

$$\|x\|_{[k]} = |x_{i_1}| + \cdots + |x_{i_k}|, \text{ where} |x_{i_1}| \geq |x_{i_2}| \geq \cdots \geq |x_{i_n}|$$

7. (5.6.P19) The spectral radius $\rho(\cdot)$ is a nonnegative, continuous, homogeneous function on M_n that is not a matrix norm, norm, seminorm, or pre-norm on M_n. Give examples to show that

   (a) $\rho(A) = 0$ is possible for some $A \neq 0$,

   (b) $\rho(A + B) > \rho(A) + \rho(B)$ is possible, and

   (c) $\rho(AB) > \rho(A)\rho(B) > 0$ is possible.

8. (5.8.P1) Let $A \in M_n$ be nonsingular and normal. Explain why the condition number for inversion of A, with respect to the spectral norm is $\kappa(A) = \rho(A)\rho(A^{-1})$.