LECTURE 5:
NORMS FOR VECTORS AND MATRICES

Problem: Measure size of vector or matrix.

What is “small” and what is “large”?

Problem: Measure distance between vectors or matrices.

When are they “close together” or “far apart”?

Answers are given by norms.

Also: Tool to analyze convergence and stability of algorithms.

VECTOR NORMS

Definition: Let V be a vector space over a field $\mathbf{F}$ ($\mathbf{R}$ or $\mathbf{C}$).
A function $||\cdot|| : V \to \mathbf{R}$ is a vector norm if for all $x, y \in V$

(1) $\|\|x\|\| \geq 0$	nonnegative
(1a) $\|\|x\|\| = 0$ iff $x = 0$	positive
(2) $\|\|cx\|\| = \|c\|\ \|\|x\|\|$ for all $c \in \mathbf{F}$	homogeneous
(3) $\|\|x+y\|\| \leq \|\|x\|\| + \|\|y\|\|$	triangle inequality

A function not satisfying (1a) is called a vector *seminorm*.

Answers: Size of vector.

INNER PRODUCTS

Definition: Let V be a vector space over a field $\mathbf{F}$ ($\mathbf{R}$ or $\mathbf{C}$).
A function $\langle \cdot,\cdot \rangle : V \times V \to \mathbf{F}$ is an inner product if for all $x, y, z \in V$,

(1) $\langle x, x\rangle \geq 0$	nonnegative
(1a) $\langle x, x\rangle = 0$ iff $x = 0$	positive
(2) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$	additive
(3) $\langle cx, y\rangle = c\langle x, y\rangle$ for all $c \in \mathbf{F}$	homogeneous
(4) $\langle x, y\rangle = \overline{\langle y, x\rangle}$	Hermitian property

Answers: “Angle” (distance)between vectors.

CONNECTIONS BETWEEN NORM AND INNER PRODUCTS

Corollary: If $\langle \cdot,\cdot \rangle$ is an inner product, then $||x|| = (\langle x, x\rangle)^{1/2}$ is a vector norm.

Called: *Vector norm derived from an inner product.*

Satisfies parallelogram identity (Necessary and sufficient condition):

$$\frac{1}{2}(||x+y||^2 + ||x-y||^2) = ||x||^2 + ||y||^2$$

Theorem (Cauchy-Schwarz inequality):

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

We have equality iff $x = cy$ for some $c \in \mathbf{F}$ (i.e., linearly dependent)

Examples

The *Euclidean norm* (l_2) on $\mathbf{C}^n$:

$$||x||_2 = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{1/2}.$$

The *sum norm* (l_1), also called one-norm or Manhattan norm:

$$||x||_1 = |x_1| + |x_2| + \cdots + |x_n|.$$

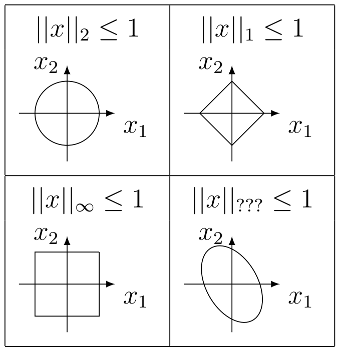
The *max norm* (l_∞):

$$||x||_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

Only Euclidean norm derived from inner product.

Unit balls for different norms

Shape of unit ball characterizes the norm.



Properties: Convex and compact (for finite dimension) around origin.

Examples cont'd

The l_p-norm on $\mathbf{C}^n$ is ($p \geq 1$):

$$||x||_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$$

Norms may also be constructed from others, e.g.,:

$$||x|| = \max\{||x||_{p_1}, ||x||_{p_2}\}$$

or let nonsingular $T \in M_n$ and $||\cdot||$ be a given, then

$$||x||_T = ||Tx||.$$

Norms on infinite-dimensional vector spaces
(e.g., all continuous functions on an interval $[a, b]$):
"similarly" defined (sums become integrals)

Convergence

Assume: Vector space V over $\mathbf{R}$ or $\mathbf{C}$.

Definition: The sequence $\{x^{(k)}\}$ of vectors in V converges to $x \in V$ with respect to $||\cdot||$ iff

$$||x^{(k)} - x|| \to 0 \quad \text{as} \quad k \to \infty.$$

Infinite dimension:

- Sequence can converge in one norm, but not another.
- Important to state choice of norm.

Convergence: Finite dimension

Corollary: For any vector norms $||\cdot||_\alpha$ and $||\cdot||_\beta$ on a finite-dimensional V, there exists $0 \le C_m < C_M < \infty$ such that

$$C_m||x||_\alpha \le ||x||_\beta \le C_M||x||_\alpha \qquad \forall x \in V$$

Conclusion: Convergence in one norm $\Rightarrow$ convergence in all norms.

Note: Result also holds for *pre-norms*, without the triangle inequality.

Definition: Two norms are *equivalent* if convergence in one of the norms always implies convergence in the other.

Conclusion: All norms are equivalent in the finite dimensional case.

Convergence: Cauchy sequence

Definition: A sequence $\{x^{(k)}\}$ in V is a Cauchy sequence with respect to $||\cdot||$ if for every $\epsilon > 0$ there is a $N_\epsilon > 0$ such that

$$||x^{(k_1)} - x^{(k_2)}|| \le \epsilon$$

for all $k_1, k_2 \ge N_\epsilon$.

Theorem: A sequence $\{x^{(k)}\}$ in a finite dimensional V converges to a vector in V iff it is a Cauchy sequence.

Dual norms

Definition: The dual norm of $\|\cdot\|$ is

$$\|y\|^D = \max_{x:\|x\|=1} \operatorname{Re} y^*x = \max_{x:\|x\|=1} |y^*x| = \max_{x\neq 0} \frac{|y^*x|}{\|x\|}$$

Examples:

Norm	Dual norm
$\|\cdot\|_2$	$\|\cdot\|_2$
$\|\cdot\|_1$	$\|\cdot\|_\infty$
$\|\cdot\|_\infty$	$\|\cdot\|_1$

- Dual of dual norm is the original norm.
- Euclidean norm is its own dual.
- Generalized Cauchy-Schwarz: $|y^*x| \le \|x\|\|y\|^D$

Vector norms applied to matrices

M_n is a vector space (of dimension n^2)

Conclusion: We can apply vector norms to matrices.

Examples:

The l_1 norm: $||A||_1 = \sum_{i,j} |a_{ij}|$.

The l_2 norm (Euclidean/Frobenius norm): $||A||_2 = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$.

The l_∞ norm: $||A||_\infty = \max_{i,j} |a_{ij}|$.

Observation: Matrices have certain properties (e.g., multiplication). May be useful to define particular *matrix norms*.

Matrix norm axioms

Definition: $|||\cdot||| : M_n \to \mathbf{R}$ is a matrix norm if for all $A, B \in M_n$,

(1)	$|||A||| \geq 0$	nonnegative
(1a)	$|||A||| = 0$ iff $A = 0$	positive
(2)	$|||cA||| = |c|\ |||A|||$ for all $c \in \mathbf{C}$	homogeneous
(3)	$|||A + B||| \leq |||A||| + |||B|||$	triangle inequality
(4)	$|||AB||| \leq |||A|||\ |||B|||$	submultiplicative

Observations:

- All vector norms satisfy (1)-(3), some may satisfy (4).
- *Generalized matrix norm* if not satisfying (4).

Which vector norms are matrix norms?

$||A||_1$ and $||A||_2$ are matrix norms.

$||A||_\infty$ is not a matrix norm (but a generalized matrix norm).

However, $|||A||| = n||A||_\infty$ is a matrix norm.

Induced matrix norms

Definition: Let $||\cdot||$ be a vector norm on $\mathbf{C}^n$. The matrix norm

$$|||A||| = \max_{||x||=1} ||Ax||$$

is *induced* by $||\cdot||$.

Properties of induced norms $|||\cdot|||$:

- $|||I||| = 1$.
- The only matrix norm $N(A)$ with $N(A) \leq |||A|||$ for all $A \in M_n$ is $N(\cdot) = |||\cdot|||$.

Last property called *minimal matrix norm*.

Examples

The maximum column sum (induced by l_1):

$$|||A|||_1 = \max_j \sum_i |a_{ij}|$$

The spectral norm (induced by l_2):

$$|||A|||_2 = \max\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\}$$

The maximum row sum (induced by l_∞):

$$|||A|||_\infty = \max_i \sum_j |a_{ij}|$$

Application: Computing Spectral radius

Recall: Spectral radius: $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

Not a matrix norm, but very related.

Theorem: For any matrix norm $|||\cdot|||$ and $A \in M_n$,

$$\rho(A) \leq |||A|||.$$

Lemma: For any $A \in M_n$ and $\epsilon > 0$, there is $|||\cdot|||$ such that

$$\rho(A) \leq |||A||| \leq \rho(A) + \epsilon$$

Corollary: For any matrix norm $|||\cdot|||$ and $A \in M_n$,

$$\rho(A) = \lim_{k\to\infty} |||A^k|||^{1/k}$$

Application: Convergence of A^k

Lemma: If there is a matrix norm with $|||A||| < 1$ then $\lim_{k\to\infty} A^k = 0$.

Theorem: $\lim_{k\to\infty} A^k = 0$ iff $\rho(A) < 1$.

Matrix extension of $\lim_{k\to\infty} x^k = 0$ iff $|x| < 1$.

Application: Power series

Theorem: $\sum_{k=0}^{\infty} a_k A^k$ converges if there is a matrix norm such that $\sum_{k=0}^{\infty} |a_k| \, |||A|||^k$ converges.

Corollary: If $|||A||| < 1$ for some matrix norm, then $I - A$ is invertible and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

Matrix extension of $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$.

Useful to compute "error" between A^{-1} and $(A+E)^{-1}$.

Unitarily invariant and condition number

Definition: A matrix norm is unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in M_n$ and all unitary matrices $U, V \in M_n$.

Examples: Frobenius norm $||\cdot||_2$ and spectral norm $|||\cdot|||_2$.

Definition: *Condition number for matrix inversion* with respect to the matrix norm $|||\cdot|||$ of nonsingular $A \in M_n$ is

$$\kappa(A) = |||A^{-1}||| \, |||A|||$$

Frequently used in perturbation analysis in numerical linear algebra.

Observation: $\kappa(A) \geq 1$ (from submultiplicative property).

Observation: For unitarily invariant norms: $\kappa(UAV) = \kappa(A)$.