Matrix Algebra
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Homework # 6

Numbers below refer to problems in Horn, Johnson "Matrix analysis." A number 1.1.P2 means Problem 2 in Section 1.1.

1. (7.1.P1) Let $A = [a_{ij}] \in M_n$ be psd. Why is $a_{ii}a_{jj} \geq |a_{ij}|^2$ for all distinct $i, j \in \{1, \ldots, n\}$? If A is pd, why is $a_{ii}a_{jj} > |a_{ij}|^2$ for all distinct $i, j \in \{1, \ldots, n\}$? If there is a pair of distinct indices i, j such that $a_{ii}a_{jj} = |a_{ij}|^2$, why is A singular?

2. (7.2.P5)

   (a) Verify that $L_1 = \begin{bmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{bmatrix}$ provides the Cholesky factorization of the pd matrix $A_1 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, and that $4 \cdot 4 \geq 2^2 \cdot (\sqrt{3})^2 = \det A_1$.

   (b) Let $A = [a_{ij}] \in M_n$ be pd and let $A = LL^*$ be a Cholesky factorization. Let $L = [c_{ij}]$ such that $c_{ij} = 0$ for $j > i$. Show that $\det A = \prod_{i=1}^n c_{ii}^2$. Show that each $a_{ii} = |c_{i1}|^2 + \ldots + |c_{i,i-1}|^2 + c_{ii}^2 \geq c_{ii}^2$, with equality iff $c_{ik} = 0$ for all $k = 1, \ldots, i-1$. Deduce Hadamard's inequality $\det A \leq \prod_{i=1}^n a_{ii}$ with equality iff A is diagonal.

3. (7.3.P7 new and old) Let $A \in M_{m,n}$ and let $A = V\Sigma W^*$ be a singular value decomposition. Define $A^\dagger = W\Sigma^\dagger V^*$, in which $\Sigma^\dagger$ is obtained from Σ by first replacing each nonzero singular value with its inverse and then transposing. Show that:

   (a) $AA^\dagger$ and $A^\dagger A$ are Hermitian

   (b) $AA^\dagger A = A$

   (c) $A^\dagger AA^\dagger = A^\dagger$

   (d) $A^\dagger = A^{-1}$ if A is square and nonsingular

   (e) $(A^\dagger)^\dagger = A$

   (f) $A^\dagger$ is uniquely determined by the properties (a)-(c)

   The matrix $A^\dagger$ is the *Moore-Penrose generalized or pseudo inverse* of A.

4. (7.3.P10) Let $A = V\Sigma W^*$ be a singular value decomposition of $A \in M_{m,n}$ and let $r = \operatorname{rank} A$. Show that:

   (a) The last $n-r$ columns of W are an orthonormal basis for the null space of A.

   (b) The first r columns of V are an orthonormal basis for the range of A.

   (c) The last $n-r$ columns of V are an orthonormal basis for the null space of A^*.

   (d) The first r columns of W are an orthonormal basis for the range of A^*.

5. We know that if A and B are pd then $A \circ B$ is pd. Show that $A \circ B$ can be pd even if not both A and B are pd.

6. (7.7.P14) Let $A, B \in M_n$ be pd and let $\alpha \in (0,1)$. Show that $\alpha A^{-1} + (1-\alpha)B^{-1} \geq (\alpha A + (1-\alpha)B)^{-1}$, with equality iff $A = B$. Thus the function $f(t) = t^{-1}$ is strictly convex on the set of pd matrices.

7. (7.8.P12, similar to 7.8.P21 in old edition) Let $A = [a_{ij}] \in M_n$ be pd. Partition $A = \begin{bmatrix} A_{11} & x \\ x^* & a_{nn} \end{bmatrix}$, in which $A_{11} \in M_{n-1}$. Use the Cauchy expansion (0.8.5.10) or the Schur complement to show that $\det A = (a_{nn} - x^* A_{11}^{-1} x) \det A_{11} \leq a_{nn} \det A_{11}$, with equality iff $x = 0$. Use this observation to give a proof by induction of Hadamard's inequality (7.8.2) and its case of equality.