

- Field of values
- **Stable matrices**
- (SVD)



Parts of Chapter 1, 2, and (3) of "Topics in Matrix Analysis," by R. A. Horn and C. R. Johnson + additional material (e.g., from "Matrices - Methods and Applications" by S. Barnett).

## FIELD OF VALUES, CONT'D

Can be useful to derive conditions on eigenvalues, e.g., in stability analysis.

Example: If  $A + A^*$  is positive definite, then  $A$  is positive stable (i.e., all its eigenvalues are in the right half plane of  $\mathbb{C}$ .)



## THE FIELD OF VALUES

The field of values (or the numerical range),  $F(A)$ , is a set of complex numbers associated with a matrix  $A \in M_n$ :

$$F(A) \equiv \{x^* A x : x \in \mathbb{C}^n, x^* x = 1\}$$

The spectrum of  $A$  is a discrete point set while the field of values can be a continuum.  $F(A)$  is always a compact convex set.



Clearly,

$$F(A + B) \subset F(A) + F(B) \quad (\text{sum of all possible element pairs})$$

$$\sigma(A) \subset F(A)$$

## STABLE MATRICES: INTROD.

System of  $n$  first order differential equations:

$$\frac{dx(t)}{dt} = A(x(t) - \hat{x}); \quad x(0) = x_0$$

Solution:

$$x(t) = e^{At}(x_0 - \hat{x}) + \hat{x}$$

where

$$e^{At} \equiv \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$



Fact:  $\hat{x}$  is a globally stable equilibrium iff the eigenvalues of  $A$  have negative real parts.

For nonlinear differential equations we may check local stability by studying the linearized system (sufficient conditions).

## STABLE MATRICES: INERTIA FOR COMPLEX MATRICES

Def: Let  $A \in M_n$ .

$i_+(A)$  is the number (with multiplicities) of eigenvalues of  $A$  with positive real part.

$i_-(A)$  is the number (with multiplicities) of eigenvalues of  $A$  with negative real part.

$i_0(A)$  is the number (with multiplicities) of eigenvalues of  $A$  with zero real part.

Clearly,  $i_+(A) + i_-(A) + i_0(A) = n$ . We also define the *inertia* of  $A$  as

$$i(A) = [i_+(A), i_-(A), i_0(A)]$$



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## LYAPUNOV'S THEOREM

$A \in M_n$  is positive stable if and only if there exists a positive definite  $G \in M_n$  such that

$$GA + A^*G = H \quad (1)$$

is positive definite. Furthermore, if there are Hermitian matrices  $G, H \in M_n$  that satisfy (1) and suppose  $H$  is positive definite, then  $A$  is positive stable if and only if  $G$  is positive definite.



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## STABLE MATRICES

Def: A matrix  $A \in M_n$  is said to be positive stable if  $i(A) = [n, 0, 0]$ .

Clearly, if  $A$  is positive stable

- $-A$  is negative stable; and
- $A^{-1}, A^*, A^T$  are positive stable.



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### Connections: Matrices – Polynomials

- All roots of a polynomial lie in the right half-plane if and only if the corresponding companion matrix is positive stable.
- A matrix is positive stable if and only if the roots of its characteristic polynomial lie in the right half-plane.

## LYAPUNOV CONT'D

Thm: If  $A$  is positive stable, Lyapunov's equation has a unique solution  $G$  for each  $H$ . If  $H$  is Hermitian, so is  $G$  and if  $H$  is positive definite, then  $G$  is positive definite (more later).

Given a positive stable matrix  $A$ , the Lyapunov equation may be seen as a function that for each positive definite matrix  $H$  produces a (unique) positive definite matrix  $G \equiv G_A(H)$ .

One common form of the Lyapunov equation is

$$GA + A^*G = I$$

Test for stability: Solve for  $G$  (Hermitian) and check if it is positive definite.

## LYAPUNOV: THE DISCRETE TIME CASE

For the difference equation:

$$x(k+1) = Ax(k)$$

we know it is asymptotically stable if and only if all eigenvalues of  $A$  are inside the unit circle.

Construct the Lyapunov function  $V(k) = x^*(k)Gx(k)$  where  $G > 0$  (positive definite) and study

$$\begin{aligned} V(k+1) - V(k) &= x^*(k+1)Gx(k+1) - x^*(k)Gx(k) \\ &= x^*(k)(A^*GA - G)x(k) \end{aligned}$$

Hence, if  $(A^*GA - G) < 0$ , then  $V(k)$  will decay towards zero which implies that  $\|x(k)\| \rightarrow 0$  as  $k \rightarrow \infty$ .



## LYAPUNOV: GENERALIZATION

Def: A matrix  $A \in M_n$  is *positive semistable* if  $v_-(A) = 0$ .

Thm: If  $GA + A^*G = H$  for some positive definite  $G$  and some positive semidefinite  $H$ , then  $A$  is positive semistable.



## DISCRETE TIME LYAPUNOV THEOREM

The matrix  $A \in M_n$  is convergent ( $|\lambda_i(A)| < 1$ ) if and only if there exist positive definite matrices  $G, H \in M_n$  such that

$$A^*GA - G = -H$$



## LYAPUNOV THEOREM CONT'D

If  $H$  in the Lyapunov equations

$$\begin{aligned} GA + A^*G &= H \\ A^*GA - G &= -H \end{aligned}$$

is only positive semidefinite, then  $A$  is positive stable (or convergent, respectively) if and only if the following controllability condition holds:

$$\text{rank} \begin{bmatrix} H & A^*H & (A^*)^2H & \dots & (A^*)^{n-1}H \end{bmatrix} = n$$



## THE GENERAL INERTIA THEOREM

Let  $A \in M_n$  be given. There exists a Hermitian  $G \in M_n$  and a positive definite  $H \in M_n$  such that

$$GA + A^*G = H$$

if and only if  $i_0(A) = 0$ . In this event,  $i(A) = i(G)$ .



## ROUTH-HURWITZ, CONT'D

One can show that the Routh-Hurwitz test is equivalent to checking the signs of the diagonal elements in a triangularized version of the Routh-Hurwitz matrix (see, e.g., Barnett "Matrices: Methods and Applications"). This leads to the so called Routh array:

Define ( $a_0 = 1$ ):

$$\begin{aligned} \{r_{01}, r_{02}, r_{03}, \dots\} &= \{a_0, a_2, a_4, \dots\} \\ \{r_{11}, r_{12}, r_{13}, \dots\} &= \{a_1, a_3, a_5, \dots\} \end{aligned}$$

Each subsequent row is then obtained from the preceding two by:

$$r_{ij} = \frac{-\det \begin{bmatrix} r_{i-2,1} & r_{i-2,j+1} \\ r_{i-1,1} & r_{i-1,j+1} \end{bmatrix}}{r_{i-1,1}}; \quad i = 2, 3, \dots, n$$



## ROUTH-HURWITZ CONDITIONS

Let  $A \in M_n(\mathbf{R})$  and  $p_A(t) = \det(tI - A) = t^n + a_{11}t^{n-1} + \dots + a_n$ . Construct the Routh-Hurwitz matrix:

$$\Omega(A) = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & a_{2n-1} \\ 1 & a_2 & a_4 & \dots & a_{2n-2} \\ 0 & a_1 & a_3 & \dots & a_{2n-3} \\ 0 & 1 & a_2 & \dots & a_{2n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

where  $a_r = 0, r > n$ .

Thm: The matrix  $A$  is negative stable if and only if the leading principal minors of  $\Omega(A)$  are positive.



## ROUTH-HURWITZ, CONT'D

Routh's thm: The polynomial  $p_A(t)$  has all roots in the left half plane if and only if all elements  $r_{i1}$  in the first column are positive.

A similar test has been developed for checking if roots are inside the unit circle (Schur-Cohn).

We may also use the bilinear transformations

$$\mu = \frac{\lambda + 1}{\lambda - 1} \quad \lambda = \frac{\mu + 1}{\mu - 1}$$

which is a one-to-one mapping between the left half of the  $\lambda$ -plane to the unit disc

$$|\mu| < 1.$$

By this we have

$$a(\lambda) = a\left(\frac{\mu + 1}{\mu - 1}\right) = \frac{b(\mu)}{(\mu - 1)^n}$$

and, hence, to check whether  $b(\mu)$  has roots inside the unit circle is equivalent to check if  $a(\lambda)$  has roots in the left half plane or vice versa.



Chapter 3 begins with a review of the singular value decomposition partially overlapping with the first book. It may be fun to read the introductory section about historical remarks. For the ones working with the SVD or whenever you need properties (bounds) on singular values, this is a good source of information.