Homework # 8

Numbers below refer to problems in Horn, Johnson "Matrix analysis." A number 1.1.P2 means Problem 2 in Section 1.1.

1. (6.1.P9) Suppose that $A = [a_{ij}] \in M_n$ is strictly diagonally dominant and let $D = \text{diag}\{a_{11}, \ldots, a_{nn}\}$. Explain why D is nonsingular and show that $\rho(I - D^{-1}A) < 1$.

2. Which of the matrices

$$A = \begin{bmatrix} 5 & -2 & 0 & 3 & 0 \\ 1 & 4 & -1 & 1 & -1 \\ 2 & 0 & 5 & 1 & 2 \\ -3 & 0 & 0 & 3 & 0 \\ 3 & -1 & -2 & 1 & 7 \end{bmatrix}, \ B = \begin{bmatrix} 5 & 0 & 0 & 3 & 0 \\ 1 & 4 & -1 & 1 & -1 \\ 2 & 0 & 5 & 1 & 2 \\ -1 & 0 & 0 & 3 & 0 \\ 2 & -1 & -2 & 1 & 7 \end{bmatrix}, \ C = \begin{bmatrix} 5 & -2 & 0 & 3 & 0 \\ 1 & 4 & -1 & 1 & -1 \\ 2 & 0 & 5 & 1 & 2 \\ -3 & 0 & 0 & 3 & 0 \\ 2 & 0 & -2 & 1 & 7 \end{bmatrix}$$

   are

   a) irreducible?
   b) diagonally dominant?
   c) irreducibly diagonally dominant?

3. (6.3.P7, 6.3.P3 in the old edition) Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $A + tE$ for $t > 0$.

   (a) Does A satisfy the hypothesis of Theorem 6.3.12?
   (b) Show that the eigenvalues of $A + tE$ are $\pm\sqrt{t}$, and explain why the eigenvalue $\lambda(t) = \sqrt{t}$ is continuous but not differentiable at $t = 0$.
   (c) Does A satisfy the hypotheses of Theorem 6.3.2?
   (d) Let λ be an eigenvalue of A and let $\lambda(t)$ be an eigenvalue of $A + tE$. Explain why there is no c such that $|\lambda(t) - \lambda| \le c|||E|||$ for all $t > 0$ and contrast with the bound (6.3.3).

4. (6.3.P6 in the old edition) Consider Given's example of a real symmetric matrix 2-by-2 matrix $A = I$ and a real symmetric perturbation

$$E(\epsilon) = \begin{bmatrix} \epsilon\cos(2/\epsilon) & \epsilon\sin(2/\epsilon) \\ \epsilon\sin(2/\epsilon) & -\epsilon\cos(2/\epsilon) \end{bmatrix}, \ \epsilon > 0$$

with $E(0) = \lim_{\epsilon\to 0} E(\epsilon) = 0$. Show that the eigenvalues of $A + E(\epsilon)$ are $1+\epsilon$ and $1-\epsilon$, and that the respective (uniquely determined up to sign) normalized real eigenvectors are

$$x_1 = \begin{bmatrix} \cos(1/\epsilon) \\ \sin(1/\epsilon) \end{bmatrix}, \quad x_2 = \begin{bmatrix} \sin(1/\epsilon) \\ -\cos(1/\epsilon) \end{bmatrix}$$

for $\epsilon > 0$. Show that as $\epsilon \to 0$, each eigenvector points in any given direction infinitely often. Thus, even if we restrict our attention to real symmetric matrices, an individual eigenvector may vary rapidly if its eigenvalue is not well separated from others.

5. (8.1.P1) If $A \in M_n$ is nonnegative and if A^k is positive for some positive integer k, explain why $\rho(A) > 0$.

6. (8.3.P8, 8.2.P8 in the old edition) Let $A \in M_n$ be nonnegative. Show that either A is irreducible or there is a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ \mathbf{0} & & A_k \end{bmatrix}$$

is block upper triangular, and each diagonal block is irreducible (possibly a 1-by-1 zero matrix). This is an *irreducible normal form* (*Frobenius normal form*) of A. Observe that $\sigma(A) = \sigma(A_1) \cup \cdots \cup \sigma(A_k)$ including multiplicities, so the eigenvalues of a nonnegative matrix are zero (with arbitrary multiplicity) together with the spectra of finitely many nonnegative nonzero irreducible matrices. An irreducible normal form is not necessarily unique.

7. (8.4.P11) Let $n > 1$ be a prime number. If $A \in M_n$ is irreducible, nonnegative and nonsingular, explain why either $\rho(A)$ is the only eigenvalue of A of maximum modulus or all the eigenvalues of A have maximum modulus.

8. (8.7.P3, 8.7.P1 in the old edition) Let $A \in M_n$ be a nonnegative nonzero matrix that has a positive eigenvector $x = [x_i]$ and let $D = \mathrm{diag}\{x_1, \ldots, x_n\}$. Show that $\rho(A)^{-1} D^{-1} A D$ is stochastic. This observation permits many questions about nonnegative matrices with a positive eigenvector to be reduced to questions about stochastic matrices.