REGLERTEKNIK

School of Electrical Engineering, KTH

EL2520 Control Theory and Practice – Advanced Course

Exam (tentamen) 2013–05–25, kl 09.00–14.00

Aids: The course book for EL2520 (advanced course) and EL1000/EL1100 (basic course), copies of slides from this year's lectures, mathematical tables and pocket calculator. Note that exercise materials (övningsuppgifter, ex-tentor och lösningar) are NOT allowed.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be motivated.
Unjustified answers will results in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Please use only one side of each sheet.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have been marked.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

Responsible: Mikael Johansson 08-7907436

Results: Will be posted no later than June 15, 2013.

Good Luck!

1. (a) Consider the multivariable linear system

$$\dot{x} = \begin{bmatrix} -\alpha - 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} \alpha/2 - 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u$$

Determine the corresponding transfer matrix and calculate the poles and zeros of the system. Characterize for which values α the system is stable, and for which values of α the system is minimum phase. (4p)

(b) For which values of α is the system observable? (2p)

(c) Given the system

$$G(s) = \frac{1}{s+1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Compute the singular values of $G(i\omega)$ and determine $\|G\|_\infty$ (4p)

2. You have been asked to design a controller for a system G with state-space realization

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

(a) Initially, you were told that there were no disturbances acting on the system, and that the full state vector x could be measured. You decided to design an optimal LQ controller and use a criterion on the form

$$J = \int x_2(t)^T Q_1 x_2(t) + u(t)^T Q_2 u(t) \, \mathrm{d}t$$

You consider four weight choices

$$\begin{array}{ll} \mathbf{A}: Q_1 = 1,\ Q_2 = 1 & \mathbf{B}: Q_1 = 1,\ Q_2 = 0.01 \\ \mathbf{C}: Q_1 = 0.01,\ Q_2 = 1 & \mathbf{D}: Q_1 = 1,\ Q_2 = 100 \end{array}$$

The corresponding responses to the same initial condition are shown in Figure 1. Pair the responses to the weight choices above. Justify your answer! (3p)

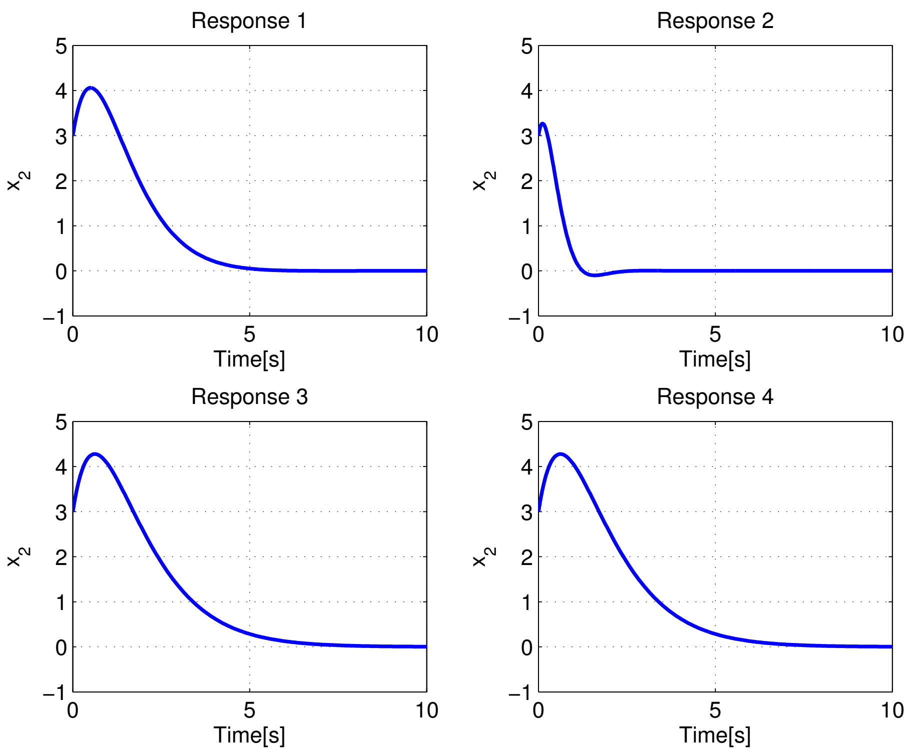

Figure 1: Initial value responses of the closed-loop system under LQ-optimal control for the four different choices of weight matrices.

(b) A few minutes after you have finalized the tuning of your LQ-controller, you learn that it will not be able to measure both system states, but that you will have to base your controller on a noisy measurement of the second system state,

$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t) + v_2(t)$$

You conclude that the best solution is probably to design an output feedback controller using LQG attempting to minimize the expected value of J. Clearly, you will need to design the Kalman filter, but what about the feedback gains? Will you need to redesign them as well? Justify your answer! (2p)

(c) Since you will need to use an observer-based controller, you decide to also try to deal with a load disturbance acting on the system output. As the disturbance has a distinct frequency content, you decide that it will be best to introduce a disturbance model as in Figure 2.

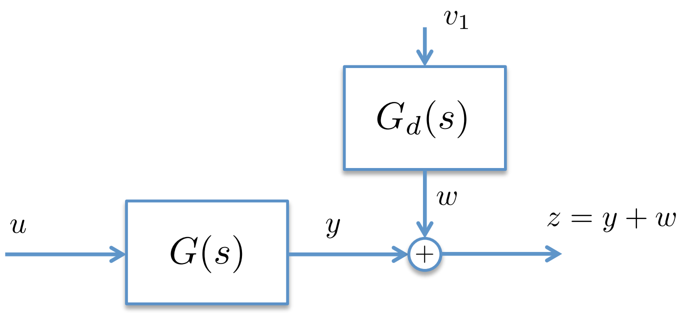

Figure 2: Augmented system with output disturbance filter $G_d(s)$.

To mimic the actual disturbance, you consider three filters:

$$\mathbf{A}: G_d(s) = \frac{1}{10s+1} \quad \mathbf{B}: G_d(s) = \frac{10}{s+10} \quad \mathbf{C}: G_d(s) = \frac{1}{s^2+0.2s+1}$$

The corresponding w for a zero-mean unit-variance white-noise v_1 are shown in Figure 3. Which realization corresponds to which filter? Justify your answer! (3p)

(d) Assume that the disturbance filter that you choose has a state-space realization

$$\begin{aligned} \dot{x}_d &= A_d x_d + N_d v_1 \\ w &= C_d x_d \end{aligned}$$

Derive a state-space model for the complete open-loop system (including the system G and the disturbance model G_d) with u and the white-noise signal v_1 as input, and $y + w$ as output.

Which is the order of the corresponding LQG-optimal controller? (2p)

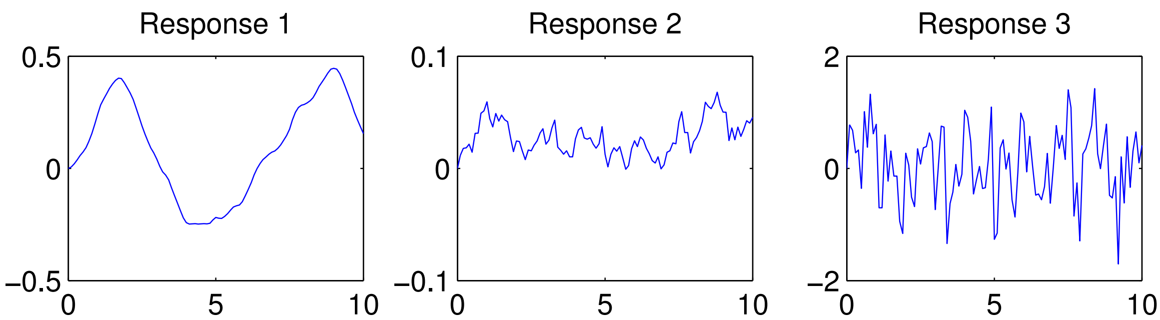

Figure 3: Realizations of the outputs of the three disturbance filters when driven by a zero-mean unit-variance input.

3. In this problem, we will study various techniques for decoupling a two-by-two system

$$Y(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} U(s) = \frac{1}{s+1} \begin{pmatrix} 1 & K_{12} \\ K_{21} & 1 \end{pmatrix} U(s)$$

(a) Compute an ideal decoupling $D(s)$ so that

$$U(s) = D(s)V(s)$$

ensures that

$$Y(s) = V(s)$$

Explain why this decoupling is not realizable for our example system (2p)

(b) When the ideal decoupling matrix is not implementable, we have recommended to use a static decoupling that makes $y(t)$ equal to $v(t)$ in stationarity.

Another possibility is to use a decoupling that retains some of the dynamics from $V(s)$ to $Y(s)$, i.e., compute a $D_1(s)$ such that

$$U(s) = D_1(s)V(s)$$

ensures that

$$Y(s) = Q(s)V(s) \tag{1}$$

for some diagonal matrix $Q(s)$. One natural choice is to let

$$Q(s) = \begin{pmatrix} G_{11}(s) & 0 \\ 0 & G_{22}(s) \end{pmatrix} \tag{2}$$

Compute $D_1(s)$ that guarantees (1) for the particular choice of $Q(s)$ stated in (2). Is the decoupling $D_1(s)$ realizable for our system? (3p)

(c) The decouplings considered in (a) and (b) are of feed-forward type, but one can also decouple systems using feedback. One such technique is called "inverted decoupling" and takes the form

$$U(s) = V(s) + \underbrace{\begin{pmatrix} 0 & \widehat{D}_{12}(s) \\ \widehat{D}_{21}(s) & 0 \end{pmatrix}}_{\widehat{D}(s)} U(s) = V(s) + \widehat{D}(s)U(s)$$

Determine the values of $\widehat{D}_{12}(s)$ and $\widehat{D}_{21}(s)$ which ensure that

$$Y(s) = \begin{pmatrix} G_{11}(s) & 0 \\ 0 & G_{22}(s) \end{pmatrix} V(s)$$

(3p)

(d) When we apply the inverted decoupling to our example system, the elements of $\widehat{D}(s)$ become static gains. In this case, the feedback loop defining the compensator is no longer well-defined (There is no dynamics in the loop, but u and v satisfy a static algebraic relationship that has to be solved).

One might then be tempted to consider the modified compensator

$$U(s) = \frac{1}{sT+1}\left(V(s) + \begin{pmatrix} 0 & \widehat{D}_{12} \\ \widehat{D}_{21} & 0 \end{pmatrix} U(s)\right)$$

For what combinations of T, $\widehat{D}_{12}$ and $\widehat{D}_{21}$ will the compensator be stable? Justify your answer! (2p)

4. A multivariable system is described by the model

$$Y(s) = \underbrace{\frac{1}{10s+1}\begin{pmatrix} 1-s & 1 \\ 2 & 1 \end{pmatrix}}_{G(s)} U(s) + \underbrace{\frac{1}{10s+1}\begin{pmatrix} 4 \\ 7 \end{pmatrix}}_{G_d(s)} D(s)$$

The system shall be controlled with the aim of keeping $|y| < 1$ in the presence of disturbances with $|d| < 1$ for all frequencies ω.

(a) Determine if $G(s)$ has any fundamental bandwidth limitations (2p)

(b) Use RGA to determine which pairing of inputs and outputs that should be used in the case of decentralized control. The desired bandwidth is around 1 rad/s. (3p)

(c) It turns out that the system is uncertain, and the uncertainty can be described by the model

$$G_p = G(I + \Delta_I)$$

for some Δ_I with $|\Delta_I(i\omega)| < |w_I(i\omega)|$ for all ω. In addition to the disturbance rejection requirement, you would also like to ensure robust stability of the closed-loop system.

Find weighted transfer functions M_1 and M_2 such that the disturbance rejection and robust stability requirements can be expressed as

$$\|M_1\|_\infty < 1$$
$$\|M_2\|_\infty < 1$$

(5p)

5. We are given a system with a non-minimum phase zero for $s = T$:

$$Y(s) = G(s)U(s) \quad \text{with} \quad G(s) = \frac{T - s}{1 + s}.$$

How should the proportional gain K_p in

$$U(s) = -K_p Y(s) \tag{3}$$

be chosen so that the quantity

$$J = \int y^2(t)\, dt$$

is kept to its minimum?

We will try to address this question using linear-quadratic control theory.

(a) Derive a state-space realization of $G(s)$ on the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Explain how you, since C is invertible, can assume access to the system state x based on a noise-free measurement of y and knowledge of the control signal u without constructing a standard observer. (2p)

Hint. Since the process has a direct term, it might be useful to observe that one can re-write the system on the form

$$\begin{aligned} Y_1(s) &= G_1(s)U(s) \\ Y(s) &= Y_1(s) + DU(s) \end{aligned}$$

for an appropriate strictly proper transfer function $G_1(s)$.

(b) Consider the LQ-criterion

$$\int y^2(t) + \rho u^2(t)\ dt.$$

Derive explicit expressions for the matrices Q_1, Q_{12} and Q_2 (in terms of ρ and the system matrices A, B, C and D) so that this criterion can be re-written as

$$\int z^T(t) Q_1 z(t) + 2x^T Q_{12} u(t) + u^T Q_2 u(t)$$

where $z(t) = Mx(t) = x(t)$. (2p)

(c) Derive the optimal state feedback law

$$u(t) = -Lx(t)$$

that minimizes the criterion in (b). Derive an explicit expression for where the closed-loop pole (under the optimal state feedback) is located as $\rho \to 0$. (5p)

Hint. Note that you do not have to characterize the closed-loop pole location explicitly for *all* ρ, only for $\rho \to 0$.

(d) Use your findings in (a) and (c) to determine the optimal gain K_p in (3) (1p)

1. (a) The transfer matrix is given by

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} \dfrac{s+3\alpha/2}{s+\alpha+1} & \dfrac{\alpha/2-1}{s+\alpha+1} \\ 0 & \dfrac{1}{s+1} \end{bmatrix}$$

The poles of the system are the eigenvalues of the A matrix. Since the A matrix is diagonal, we immediately see that the system has poles for $s = -1$ and $s = -(\alpha+1)$. It is, of course, possible to compute the poles from the transfer matrix, but the result will be the same.

As for the zeros, the maximal minor is the determinant of $G(s)$, and since

$$\det G(s) = \frac{s+3\alpha/2}{(s+1)(s+\alpha+1)}$$

the system has a zero for $s = -3\alpha/2$.

The system is stable when all poles are in the left half plane, *i.e.* when $\alpha > -1$. The system zero will be minimum phase if it is in the left half plane, *i.e.* if $\alpha > 0$ (the system will be non-minimum phase if $\alpha < 0$. Note that when $\alpha < -1$, the system is both unstable and has a non-minimum phase zero, and will be difficult to control.

(b) The system is observable when the observability matrix has full rank. Here,

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -(\alpha+1) & 0 \\ 0 & -1 \\ -(\alpha+1)(\alpha/2-1) & 0 \\ 0 & 1 \end{bmatrix}$$

looses rank if $\alpha = -1$. For all other values of α, the system is observable.

(c) The singular values are the square roots of the eigenvalues of $G(i\omega)G(i\omega)^*$. We get

$$G(i\omega)G(i\omega)^* = \frac{1}{i\omega+1}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\frac{1}{-i\omega+1}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \frac{1}{\omega^2+1}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

We detrmine the eigenvalues by calculating the roots of the characteristic polynomial wich is the determinant of $(\lambda I - G(i\omega)G(i\omega)^*)$. We get

$$\lambda I - G(i\omega)G(i\omega)^* = \frac{1}{\omega^2+1}\begin{bmatrix} (\omega^2+1)\lambda - 5 & -2 \\ -2 & (\omega^2+1)\lambda - 1 \end{bmatrix}$$

So we need to find the roots of

$$\lambda^2 - \frac{6}{\omega^2+1}\lambda + \frac{1}{(\omega^2+1)^2} = 0$$

Solving this yields

$$\lambda = \frac{3 \pm \sqrt{8}}{\omega^2 + 1}$$

The singular values are given as the root of this and hence they become

$$\sigma_{1,2} = \sqrt{\frac{3 \pm \sqrt{8}}{\omega^2 + 1}}$$

The norm we are looking form is given by

$$\|G\|_\infty = \sup_\omega \bar{\sigma}(G(i\omega)) = \sup_\omega \left(\sqrt{\frac{3 + \sqrt{8}}{\omega^2 + 1}} \right)$$

Since ω only appear as a square in the denominator we realize that the maximizing ω is $\omega = 0$ which yield

$$\|G\|_\infty = \sqrt{3 + \sqrt{8}}$$

2. (a) To be able to solve this problem, it is important to recall that scaling all weight matrices with the same constant does not affect the solution. In other words, the LQ-optimal controller for the criterion

$$J = \int x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t) \text{ d}t$$

is identical to the LQ-optimal controller for the criterion

$$J_2 = \kappa J = \int x(t)^T \kappa Q_1 x(t) + u(t)^T \kappa Q_2 u(t) \text{ d}t$$

If this is not immediately clear, then it is straightforward to verify that if P is the solution to the Riccati equation for J, then κP is the solution to the Riccati equation of J_2. The additional κ is cancelled in the expression for the feedback gains, implying that the two problems have the same solutions.

Normalizing the weights so that $Q_2 = 1$ for all weight choices, we have

$$\begin{array}{ll} \mathbf{A}: Q_1 = 1,\ Q_2 = 1 & \mathbf{B}: Q_1 = 100,\ Q_2 = 1 \\ \mathbf{C}: Q_1 = 0.01,\ Q_2 = 1 & \mathbf{D}: Q_1 = 0.01,\ Q_2 = 1 \end{array}$$

Using the rule that the larger the penalty on state deviations, the faster the system response, we see that the correct pairing is $A - 1$, $B - 2$ while C and D correspond to response 3 and 4.

(b) The LQG-optimal controller satisfies a separation principle. The feedback gains can be computed as if you had full access to the system state, and the estimator should be designed to minimize the estimation error variance. Hence, there is no need to re-design the feedback gains, they will be the same as you computed under the assumption of full state feedback.

(c) Disturbance filters model the frequency content of the disturbance. In this case, there is one filter , **C**, with energy concentrated at a distinct frequency (1 rad/sec); clearly, this corresponds to response 1. Then there are two first-order filters, both with unit static gains. However, filter **B** has a higher bandwidth, and hence allows more high-frequency content of v_1 to pass through. We can see that **B** corresponds to response 3, hence **A** corresponds to response 2.

(d) The augmented system model has states which represent the process and the disturbance filter. Thus

$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}_d(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A_d \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix} + \begin{pmatrix} 0 \\ N_d \end{pmatrix} v_1 + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} C & C_d \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix} + v_2$$

The order of the LQG-controller is the same as the order of the augmented system, *i.e.* the number of states of the system model plus the number of states of the disturbance model.

3. (a) $Y(s) = G(s)U(s)$ and $U(s) = D(s)V(s)$ yields

$$Y(s) = G(s)D(s)V(s)$$

so $Y(s) = V(s)$ implies that $G(s)D(s) = I$, *i.e.*

$$D(s) = G(s)^{-1} = \frac{s+1}{1-K_{12}K_{21}} \begin{pmatrix} 1 & -K_{12} \\ -K_{21} & 1 \end{pmatrix}$$

(b) Similarly, $Y(s) = G(s)U(s)$ and $U(s) = D_1(s)V(s)$ yields $Y(s) = G(s)D_1(s)V(s)$. The desire that $Y(s) = Q(s)V(s)$ is satisfied if

$$D_1(s) = G^{-1}(s)Q(s) = \frac{s+1}{1-K_{12}K_{21}} \begin{pmatrix} 1 & -K_{12} \\ -K_{21} & 1 \end{pmatrix} \frac{1}{s+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$
$$= \frac{1}{1-K_{12}K_{21}} \begin{pmatrix} 1 & -K_{12} \\ -K_{21} & 1 \end{pmatrix}$$

which we can realize (it is a constant matrix) unless $K_{12}K_{21} = 1$.

(c) $Y(s) = G(s)U(s)$ and $U(s) = V(s) + \hat{D}(s)U(s)$ implies that

$$Y(s) = G(s)(I - \hat{D}(s))^{-1}V(s) = Q(s)V(s)$$

is satisfied if $I - \hat{D}(s) = Q^{-1}(s)G(s)$, *i.e.* if

$$\hat{D}(s) = I - Q^{-1}(s)G(s) = \begin{pmatrix} 0 & -G_{12}(s)/G_{11}(s) \\ -G_{21}(s)/G_{22}(s) & 0 \end{pmatrix}$$

In other words $\hat{D}_{12}(s) = -G_{12}(s)/G_{11}(s)$, $\hat{D}_{21}(s) = -G_{21}(s)/G_{22}(s)$.

(d) The compensator dynamics is given by

$$U(s) = (I - \frac{1}{sT+1}\hat{D})^{-1}\frac{1}{sT+1}V(s) =$$
$$= \frac{1}{sT+1} \begin{pmatrix} 1 & -\hat{D}_{12}/(sT+1) \\ -\hat{D}_{21}/(sT+1) & 1 \end{pmatrix}^{-1} V(s) =$$
$$= \frac{1}{(sT+1)^2 - \hat{D}_{12}\hat{D}_{21}} \begin{pmatrix} sT+1 & \hat{D}_{12} \\ \hat{D}_{21} & sT+1 \end{pmatrix} V(s)$$

and hence stable if $T > 0$ and $\hat{D}_{12}\hat{D}_{21} < 1$.

4. (a) The system is stable, there are no time delays, and its zero is of minimum phase, so there is no apparent limitation.

(b) The RGA for zero frequency is

$$RGA(G(0)) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

As pairings never should yield negative diagonal elements, the only possible pairing is to use u_1 to control y_2 and to use u_2 to control y_1.

(c) The performance requirement is that $|S(i\omega)G_d(i\omega)| < 1$ for all frequencies. This is equivalent to $\|SG_d\|_\infty < 1$.

For robust stability, we can replace the nominal system G in the standard feedback set-up in the book by $G(I + \Delta_I)$. Standard calculations gives that the small gain theorem guarantees stability if

$$\| - (I + F_y G)^{-1} F_y G\|_\infty \|\Delta_I\|_\infty < 1$$

SInce $|\Delta_I(i\omega)| < w_I(i\omega)|$ for all ω, and since w_I is a scalar function, we have

$$| - (I + F_y G)^{-1} F_y G||\Delta_I| < | - (I + F_y G)^{-1} F_y G||w_I| = | - (I + F_y G)^{-1} F_y G w_I|$$

So, if we can keep the rightmost quantity below one for all frequencies, the closed-loop system will be robustly stable.

We have thus found our two transfer functions:

$$M_1 = SG_d$$
$$M_2 = -(I + F_y G)^{-1} F_y G w_I$$

5. (a) Noting that we can write

$$Y(s) = \frac{z-s}{1+s}U(s) = \frac{T+1}{s+1}U(s) - U(s) = Y_1(s) - U(s)$$

where

$$Y_1(s) = \frac{T+1}{s+1}U(s)$$

we can represent this system on controllable canonical form

$$\begin{aligned} \dot{x} &= -x + u \\ y_1 &= (T+1)x \end{aligned}$$

and hence, the full system has state-space representation

$$\begin{aligned} \dot{x} &= -x + u \\ y &= (T+1)x - u \end{aligned}$$

corresponding to $A = -1$, $B = 1$, $C = (T+1)$ and $D = -1$.
The state-space representation is not unique. One could, for example, also use $A = -1$, $B = (T+1)$, $C = 1$ and $D = -1$.
Irrespectively of the state-space representation, since the system is scalar,

$$x = \frac{1}{C}(-Du + y)$$

so we can obtain x directly from u and y.

(b) We have

$$\begin{aligned} \int y(t)^2 + \rho u(t)^2 \, dt &= \int (cx(t) + du(t))^2 + \rho u(t)^2 \, dt = \\ &= \int x(t)c^2x(t) + 2x(t)cdu(t) + u(t)(d^2 + \rho^2)u(t) \, dt \end{aligned}$$

Identifying $z(t) = x(t)$ we see that

$$\begin{aligned} Q_1 &= c^2 = (T+1)^2 \\ Q_{12} &= cd = -(T+1) \\ Q_2 &= d^2 + \rho = 1 + \rho \end{aligned}$$

(c) According to Equation (9.15) in the course book, we need to solve the Riccati equation

$$A^TS + SA + M^TQ_1M - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T = 0$$

for a postive definite solution S. In our case, we find

$$-S - S + (T+1)^2 - \frac{(S-(T+1))^2}{1+\rho} = 0$$

We note that $\rho \to 0$ implies that $S \to 2T$ and the optimal state feedback gain tends to

$$L = \frac{Q_{12}}{Q_2} = T - 1$$

The closed loop system is given by

$$\dot{x} = (A - BL)x = (-1 - (T-1))x = -Tx$$

Hence, the closed-loop pole tends to the negative of the zero location, $s = -T$. One can show that this observation holds generally true also for higher-order systems. If a system has non-minimum phase zeros, then to minimize the total energy in the output, one should place some poles at the zero locations or, if the zeros are non-minimum phase, at the locations of the zeros mirrored in the imaginary axis. The rest of the poles could be made arbitrary fast.

Remark. If you chose the alternative state-space representation discussed in (a), you will get $Q_1 = 1$, $Q_{12} = -1$, $Q_2 = 1 + \rho$ and the Riccati equation

$$-2S + 1 - \frac{(s(T+1)-1)^2}{1+\rho} = 0$$

whose positive solution, when $\rho \to 0$ tends to $2T/(T+1)^2$, and the corresponding feedback gain is $L = (T-1)/(T+1)$.

(d) From the results in (c) and (a), we have

$$u = -(T-1)x = -\frac{T-1}{T+2}(u+y) \Rightarrow u = -\frac{T-1}{2T}y$$

Hence, the optimal proportional gain is given by $K_p = (T-1)/2T$.