REGLERTEKNIK

School of Electrical Engineering, KTH

EL2520 Control Theory and Practice – Advanced Course

Exam (tentamen) 2013–05–25, kl 09.00–14.00

Aids: The course book for EL2520 (advanced course) and EL1000/EL1100 (basic course), copies of slides from this year's lectures, mathematical tables and pocket calculator. Note that exercise materials (övningsuppgifter, ex-tentor och lösningar) are NOT allowed.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be motivated.
Unjustified answers will results in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Please use only one side of each sheet.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have marked.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

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Resultat: Will be posted no later than June 15, 2013.

Good Luck!

1. In this problem, we will study various techniques for decoupling a two-by-two system

$$Y(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} U(s) = \frac{1}{s+1} \begin{pmatrix} 1 & K_{12} \\ K_{21} & 1 \end{pmatrix} U(s)$$

   (a) Compute an ideal decoupling $D(s)$ so that

$$U(s) = D(s)V(s)$$

   ensures that

$$Y(s) = V(s)$$

   Explain why this decoupling is not realizable for our example system (2p)

   (b) When the ideal decoupling matrix is not implementable, we have recommended to use a static decoupling that makes $y(t)$ equal to $v(t)$ in stationarity.

   Another possibility is use a decoupling that retains some of the dynamics from $V(s)$ to $Y(s)$, *i.e.* compute a $D_1(s)$ such that

$$U(s) = D_1(s)V(s)$$

   ensures that

$$Y(s) = Q(s)V(s) \tag{1}$$

   for some diagonal matrix $Q(s)$. One natural choice is to let

$$Q(s) = \begin{pmatrix} G_{11}(s) & 0 \\ 0 & G_{22}(s) \end{pmatrix} \tag{2}$$

   Compute $D_1(s)$ that guarantees (1) for the particular choice of $Q(s)$ stated in (2). Is the decoupling $D_1(s)$ realizable for our system? (3p)

   (c) The decouplings considered in (a) and (b) are of feed-forward type, but one can also decouple systems using feedback. One such technique is called "inverted decoupling" and takes the form

$$U(s) = V(s) + \underbrace{\begin{pmatrix} 0 & \widehat{D}_{12}(s) \\ \widehat{D}_{21}(s) & 0 \end{pmatrix}}_{\widehat{D}(s)} U(s) = V(s) + \widehat{D}(s)U(s)$$

   Determine the values of $\widehat{D}_{12}(s)$ and $\widehat{D}_{21}(s)$ which ensure that

$$Y(s) = \begin{pmatrix} G_{11}(s) & 0 \\ 0 & G_{22}(s) \end{pmatrix} V(s)$$

   (3p)

(d) When we apply the inverted decoupling to our example system, the elements of $\widehat{D}(s)$ become static gains. In this case, the feedback loop defining the compensator is no longer well-defined (There is no dynamics in the loop, but u and v satisfy a static algebraic relationship that has to be solved).

One might then be tempted to consider the modified compensator

$$U(s) = \frac{1}{s+\varepsilon}\left(V(s) + \begin{pmatrix} 0 & \widehat{D}_{12} \\ \widehat{D}_{21} & 0 \end{pmatrix} U(s)\right)$$

How should ε be chosen to guarantee that the compensator is stable? (2p)

2. We are given a system with a non-minimum phase zero

$$Y(s) = G(s)U(s) \quad \text{with} \quad G(s) = \frac{z - s}{1 + s}.$$

How should the proportional gain K_p in

$$U(s) = -K_p Y(s)$$

be chosen so that the quantity

$$J = \int y^2(t)\, dt$$

is kept to its minimum?

We will try to address this question using linear-quadratic control theory.

(a) Derive a state-space realization of $G(s)$ on the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Explain how you, for this system, can assume access to the system state x based on a noise-free measurement of y and knowledge of the control signal u. (2p)

(b) Consider the LQ-criterion

$$\int y^2(t) + \rho u^2(t)\, dt.$$

Derive explicit expressions for the matrices Q_1, Q_{12} and Q_2 (in terms of ρ and the system matrices A, B, C and D) so that this criterion can be re-written as

$$\int z^T(t) Q_1 z(t) + 2x^T Q_{12} u(t) + u^T Q_2 u(t)$$

where $z(t) = Mx(t) = x(t)$. (2p)

(c) Derive the optimal state feedback law

$$u(t) = -Lx(t)$$

that minimizes the criterion in (b). Derive an explicit expression for where the closed-loop pole (under the optimal state feedback) is located as $\rho \to 0$. (5p)

(d) Use your findings in (a) and (c) to compute the optimal gain K_p. (1p)

Consider the linear system $U(s) = G(s)Y(s)$ with transfer matrix

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

While the individual elements of the transfer matrix are benign, we will show that the interactions make the system hard to control.

(a) Compute the poles and zeros of $G(s)$. Can you foresee any limitations in the achievable control system performance? (4p)

(b) In which input direction will the non-minimum phase effect be most prominent? In other words, which is the input direction associated with the non-minimum phase zero found in (a) ? (2p)

(c) Consider the system $G(s)$ defined in Problem 1a. Feedforward control is used to obtain reference tracking with the regulator $F(s) = G(s)^{-1}$. The corresponding block diagram is shown in Figure 1. Is the system stable? Motivate. (1p)

Figure 1: Block diagram.

(d) Rather than the precompensator in (c), you decide to use an "approximate" inverse $U(s) = F_p\tilde{U}(s)$ with

$$F_p(s) = \frac{1-s}{(s+1)(s+3)} G^{-1}(s) = \begin{pmatrix} 1 & -2(s+1)/(s+3) \\ -1 & 1 \end{pmatrix}$$

and then a simple feedback $\tilde{U}(s) = -k(R(s) - Y(s))$. What is the maximum value of k that renders the closed-loop system stable? (3p)

3. (a) We want to use decentralized control on the system

$$G(s) = \frac{1}{s+1} \begin{pmatrix} \frac{0.6}{s+1} & -0.4 \\ 0.3 & \frac{0.6}{s+1} \end{pmatrix}, \tag{3}$$

with desired cross-over frequency $\omega_c = 10$ rad/s. Which inputs and outputs are suitable to pair? Motivate. (3p)

Hint:

$$G(i\omega_c) \approx \begin{pmatrix} -0.006 - 0.001i & -0.004 + 0.04i \\ 0.003 - 0.03i & -0.006 - 0.001i \end{pmatrix},$$

$$G(i\omega_c)^{-1} \approx \begin{pmatrix} 5 + 0.05i & 3 + 34i \\ -2.4 - 26i & 5 - 0.05i \end{pmatrix}.$$

(b) Consider the feedback loop depicted in Figure 2. Let $G(s)$ have two inputs and

Figure 2: Block diagram.

two outputs. The open loop system is defined as $L(s) = G(s)F(s)$, where

$$L(s) = \begin{pmatrix} l_{11}(s) & l_{12}(s) \\ l_{21}(s) & l_{22}(s) \end{pmatrix}, \quad G(s) = \begin{pmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{pmatrix},$$

$$F(s) = \begin{pmatrix} f_{11}(s) & f_{12}(s) \\ f_{21}(s) & f_{22}(s) \end{pmatrix}.$$

Is the following statement true?

If we use a decentralized controller with diagonal elements set to zero ($f_{11}(s) = f_{22}(s) = 0$), we need to shape the transfer functions $l_{21}(s)$ and $l_{12}(s)$ to obtain reference tracking.

Motivate your answer! (3p)

(c) Consider once again the system $G(s)$ in Equation (3) and the feedback loop in Figure 2. Assume that we are controlling the system using the regulator

$$\tilde{F}(s) = \begin{pmatrix} \tilde{f}_{11}(s) & 0 \\ 0 & \tilde{f}_{22}(s) \end{pmatrix},$$

when we get an additional control performance requirement. The requirement states that the static error in one output should not affect the static error in the other output when performing reference tracking. Derive a controller, expressed in terms of $\tilde{f}_{11}(s)$ and $\tilde{f}_{22}(s)$, that fulfills the additional requirement. (4p)

4. (a) Consider the system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

Determine the state feedback that minimizes the criterion

$$\int_0^\infty y^2(t) + u^2(t)\, dt.$$

(5p)

(b) A two-link robot can be described by a linearized model with four states and two inputs

x_1 : position of link 1
x_2 : velocity of link 1
x_3 : position of link 2 (smaller link with faster dynamics)
x_4 : velocity of link 2
u_1 : torque acting on link 1
u_2 : tourque acting on link 2.

One wants to minimize the deviation of the end effector position ($x_1 + x_3$) from a reference trajectory $r(t)$. In addition it is desired to avoid saturation of the smaller arm position. It is more important to keep u_2 small than it is to keep u_1 small. This leads to the criterion

$$\min_{u_1, u_2} \int_0^\infty [r(t) - (x_1(t) + x_3(t))]^2 + x_3(t)^2 + 0.1u_1^2(t) + u_2^2(t)\, dt \qquad (4)$$

Show that this criterion can be minimized by minimizing

$$\min_u \int_0^\infty x(t)^T Q_1 x(t) + h(t)^T x(t) + u(t)^T Q_2 u(t)\, dt.$$

Determine the associated matrices Q_1, $h(t)$ and Q_2. (5p)

5. Consider the plant

$$G(s) = 12\frac{s-1}{(s+2)(s+3)}$$

and the disturbance model

$$G_d(s) = \frac{100}{s+100}$$

with their respective Bode diagrams shown in Figure 4.

Figure 3: Block diagram.

(a) Is it possible to find a controller $F(s)$ that renders the closed-loop in Figure 3 internally stable and ensures that the following specifications are met?

a1) Low-frequency disturbances $d(t)$ should be attenuated by a factor of 5 for frequencies below 1Hz;

a2) In steady-state, bounded sinusoidal disturbances with $|d(t)| \leq 1$ and frequencies above 10 rad/s should be attenuated with a control signal satisfying $|u(t)| \leq 1$.

Motivate your answer! (2p)

(b) After some iterations the specifications were modified to

b1) There should exist a stabilizing controller satisfying the specifications

b2) A sinusoidal disturbance at 150Hz should be attenuated by a factor of 100;

b3) Low-frequency disturbances below 0.01Hz should not be amplified;

b4) The steady-state tracking error $e(t) = r(t) - y(t)$ for a step reference should be smaller than 0.01.

Consider a weighted sensitivity design

$$\begin{array}{ll} \underset{F_y(s)}{\text{minimize}} & \gamma^2 \\ \text{subject to} & \|W_s S\|_\infty \leq \gamma^2 \end{array}$$

and translate each of the requirements b1)-b4)to a constraint on the sensitivity weight $W_S(s)$ so that the requirements are fulfilled if $\gamma^2 \leq 1$. Sketch the Bode magnitude plot of a weight that satisfies these constraints. (5p)

(c) Suppose a given controller $F(s)$ is obtained and you want to verify that

Bode_AT-eps-converted-to.pdf

Figure 4: Bode diagrams of $G(s)$ and $G_d(s)$.

c1) In stationarity, the control signal should satisfy $|u(t)| \leq 1$ for any combination of bounded sinusoidal disturbances

$$d(t) = d_0 \sin(\omega t + \varphi_d) \tag{5}$$

and bounded sinusoidal reference signals

$$r(t) = r_0 \sin(\omega t + \varphi_r) \tag{6}$$

with the same frequency $w = 0.5$rad/s and amplitudes satisfying $d_0^2 + r_0^2 \leq 2$.

Recalling that $u = G_{du}(s)d + G_{ru}(s)r$, derive a condition on the transfer functions that allow you to test that requirement c1) is satisfied for all disturbances on the form (5), (6) with $d_0^2 + r_0^2 \leq 2$. (3p) ic2-eps-converted-to.pdf

Figure 5: Uncertainty model.

6. (a) Compute the energy-gain (H_∞-norm) of the transfer function

$$G(s) = \frac{s+1}{(s+2)^2}$$

(3p)

(b) Consider the system $Y(s) = G(s)U(s)$ where $G(s)$ is uncertain and on the form

$$G(s) = (I - W(s)\Delta(s))^{-1} G_0(s)$$

Show that $G(s)$ can be represented as in Figure 5 for the appropriate transfer matrix $M(s)$. Assume that $U(s) = -Y(s)$ and use the small-gain theorem to derive a condition for closed-loop stability for all $\Delta(s)$ with $\|\Delta\|_\infty < d_{\max}$. (3p)

(c) Consider the nominal system

$$G_0(s) = \frac{1}{s+1}$$

and the uncertain system

$$G(s) = \frac{s+2}{(s+1)(s+2+\alpha)}$$

where α is an uncertain parameter. Show that the uncertainty can be represented as in (b) with $\Delta(s) = \alpha$ and determine the associated $W(s)$. (2p)

(d) Use the results from (b) and (a) to compute a bound $\alpha_{\max}$ such that the closed-loop system is stable for all α with $|\alpha| \leq \alpha_{\max}$. (1p)

(e) How does the small-gain result in (d) compare with a direct analysis that first computes the closed-loop transfer function for $G(s)$ under the feedback $U(s) = -Y(s)$ and then inspects for which α the closed-loop system is stable? (1p)

1. (a) The minors are

$$\frac{1}{s+1}, \frac{2}{s+3}, \frac{1}{s+1}, \frac{1}{s+1}, \det G(s) = \frac{1}{(s+1)^2} - \frac{2}{(s+1)(s+3)} = \frac{1-s}{(s+1)^2(s+3)}$$

Hence, the system has a double pole in $s = -1$ and an additional pole in $s = -2$, while it has a non-minimum phase zero for $s = 1$. The non-minimum phase zero limits the frequency interval over which we can have good disturbance rejection.

(b) Since

$$G(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

we can see that the output is zero for any input on the form $c(1,\ -1)$. Hence, $(1,\ -1)$ is the input direction assoicated with the zero.

(c) Since the process has a zero in the right-half plane, its inverse will be unstable. Hence, the series connection will not be internally stable.

(d) We have

$$Y(s) = G(s)U(s) = G(p)F_p\tilde{U}(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix} \begin{pmatrix} 1 & -2\frac{s+1}{s+3} \\ -1 & 1 \end{pmatrix} \tilde{U}(s) =$$
$$= \begin{pmatrix} \frac{1-s}{(s+1)(s+3)} & 0 \\ 0 & \frac{1-s}{(s+1)(s+3)} \end{pmatrix} \tilde{U}(s)$$

Since the system is decoupled and the controller is diagonal, we can consider the loops in isolation. The closed-loop from r_1 to y_1 is

$$\frac{k(1-s)}{(s+1)(s+3)+k(1-s)} = \frac{k(1-s)}{s^2+(4-k)s+3+k}$$

For stability, all coefficients of the characteristic polynomial need to be positive. Hence, the maximum value for k that ensures stability is $k = 4$.

2.

a) Use RGA-analysis.

$$RGA(G(0)) \approx \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}.$$

All elements are positive, therefore all pairings are possible. Let us look at $RGA(G(i\omega_c))$.

$$RGA(G(i\omega_c)) \approx \begin{pmatrix} -0.03 - 0.005i & 1.05 + 0.008i \\ 1.03 + 0.012i & -0.03 - 0.005i \end{pmatrix}.$$

The off-diagonal elements are close to 1, therefore u_2 should be paired with y_1 and u_1 should be paired with y_2.

b) No, the statement is false. We have the following relationship:

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} l_{11}(s) & l_{12}(s) \\ l_{21}(s) & l_{22}(s) \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \\ &= \begin{pmatrix} l_{11}(s)(r_1 - y_1) & l_{12}(s)(r_2 - y_2) \\ l_{21}(s)(r_1 - y_1) & l_{22}(s)(r_2 - y_2) \end{pmatrix}, \end{aligned}$$

where r_1 and r_2 are the reference signals. Solving for y_1 and y_2 gives

$$\begin{aligned} y_1 &= \frac{l_{11}}{1 + l_{11}} r_1 + \frac{l_{12}}{1 + l_{11}} (r_2 - y_2), \\ y_2 &= \frac{l_{21}}{1 + l_{22}} (r_1 - y_1) + \frac{l_{22}}{1 + l_{22}} r_2. \end{aligned}$$

The control objective was to perform reference tracking, that is y_1 should follow r_1 and y_2 should follow r_2. Therefore, we would like to shape

$$\begin{aligned} y_1 &= \frac{l_{11}}{1 + l_{11}} r_1 = G_{c,11} r_1, \\ y_2 &= \frac{l_{22}}{1 + l_{22}} r_2 = G_{c,22} r_2. \end{aligned}$$

We know from the theory of loop shaping that we can design the closed loop transfer function by shaping the open loop transfer function, in our case l_{11} and l_{22}. Thus, we would always like to shape l_{11} and l_{22}, independently of the decentralization of the controller (diagonal or off-diagonal). For more details, see the document labeled Additional instructions for lab 3 and lab 4 by Erik Henriksson on the course web page.

c) To decouple the system at zero frequency we use $G(0)^{-1}$ as filter in the following way

$$F(s) = W_1(s)\tilde{F}(s)W_2(s) = G^{-1}(0)\tilde{F}(s)I = \begin{pmatrix} 1.25 f_{11}(s) & 0.83 f_{22}(s) \\ -0.63 f_{11}(s) & 1.25 f_{22}(s) \end{pmatrix}.$$

See for example the computer exercise on Decoupling and Glover-McFarlane robust loop-shaping.

3. (a) The problem is matched to the standard LQ problem. $z = y$ gives $M = I$. $Q_1 = C^T C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q_2 = I$. The optimal controller is given by $L = Q_2^{-1} B^T S$, where $S \geq 0$ is the solution to

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0.$$

The solution is $S = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$ which gives $L = \begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}$.

Answer: The optimal feedback is $L = 1 \quad \sqrt{2}$.

(b) Identifying coefficients between the two expressions gives

$$Q_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix} \quad h(t) = \begin{pmatrix} -2r(t) \\ 0 \\ -2r(t) \\ 0 \end{pmatrix}$$

The difference between the two criteria is a term containing r. Because it does not depend on u nor x, it will not affect the solution to the minimization problem. The two forms are therefore equivalent.

4. a) The plant $G(s)$ has a RHP zero at $z = 1$, leading to the constraint $S(z) = 1$ for any internally stabilizing controller. For condition a1) to hold we need $|S(jw)| \leq 0.2 \quad \forall w \leq 2\pi$, which cannot be met for $w = z = 1$ due to the RHP zero.

Condition a2) requires $|G(jw)| > |G_d(jw)| \quad \forall w$ which does not hold as seen in Figure 4.

b) Consider the mixed sensitivity design objective $\|W_S(s)S(s)\|_\infty \leq 1$. First note that $G(s)$ has a RHP zero at $z = 1$ and thus condition b1) requires that $|W_S(z)| \leq 1$. Condition b2) corresponds to $|S(j300\pi)| \leq \frac{1}{100}$, which implies $|W_S(j300\pi)| \geq 100$. Condition b3) can be posed as the requirement $|S(jw)| \leq 1 \quad \forall w \leq 0.02\pi$ and the corresponding condition of the weight is $|W_S(jw)| \geq 1 \quad \forall w \leq 0.02\pi$. Recall that the transfer function from the reference $r(t)$ to the tracking error $e(t) = r(t) - y(t)$ is $G_{re}(s) = S(s)$. Thus b4) is equivalent to $|S(0)| \leq 0.01$ which leads to $|W_S(0)| \geq 100$.

c) Note that the specification concerns any given combination of bounded sinusoidal signals with the same frequency $w \leq 0.5$rad/s. Hence $u = G_{du}(s)d + G_{ru}(s)r$ can be seen as a MIMO system with one output and two inputs, $u = [G_{du}(s) \quad G_{ru}(s)][d \quad u]^\top = G_w(s)w$. As the inputs are sinusoids, the specification is equivalent to have

$$\max_{\|W(jw)\|_2 = \sqrt{2}} \frac{\|U(jw)\|_2}{\|W(jw)\|_2} = \sqrt{2}\sigma_1(G_w(jw)) \leq 1 \quad \forall w \leq 0.5\text{rad/s}$$

Hence the specification holds if and only if

$$\sigma_1([G_{du}(jw) \quad G_{ru}(jw)]) \leq \frac{1}{\sqrt{2}} \quad \forall w \leq 0.5\text{rad/s}$$

5. (a)

$$\|G\|_\infty = \sup_{\omega \geq 0} |G(i\omega)| = \frac{\sqrt{1+\omega^2}}{4+\omega^2} := f(\omega)$$

Note that $f(0) = 1/4$, $f(\omega) \to 0$ as $\omega \to \infty$. In addition, we see that since $f(w)$ is positive, it is maximized for ω that maximizes $g(\omega) = f^2(\omega)$. We find the stationary points of $g(\omega)$:

$$g'(\omega) = \frac{2\omega(4+\omega^2)^2 - (1+\omega^2)2 \cdot 2\omega(4+\omega^2)}{(4+\omega^2)^4} = \frac{2\omega(4+\omega^2)(2-\omega^2)}{(4+\omega^2)^4} = 0$$

which yields the critical frequencies $\omega = 0$ and $\omega = \sqrt{2}$. Since $f(\sqrt{2}) = \sqrt{3}/6 > 1/4$, we have that $\|G\| = \sqrt{3}/6$.

(b) Following the notation in Figure 5, we have $Y = G_0U + MY \Rightarrow Y = (I - M)^{-1}G_0U$. Hence, uncertainty model can be represented as in the figure, with $M = W(s)\Delta(s)$.

Introduce $Z(s)$ and $V(s)$ such that $V(s) = \Delta(s)Z(s)$. Then

$$Z(s) = W(s)V(s) - G_0(s)Z(s) \Rightarrow Z(s) = (I + G_0(s))^{-1}W(s)V(s)$$

The small gain theorem now gives

$$\|(I + G_0(s))^{-1}W(s)\|_\infty \|\Delta(s)\|_\infty < 1$$

Hence, to ensure stability for all $\|\Delta(s)\|_\infty \leq d_{\max}$, we must require that

$$\|(I + G_0(s))^{-1}W(s))\|_\infty \leq \frac{1}{d_{\max}}$$

(c) The uncertainty model is

$$G(s) = (I - W(s)\Delta(s))^{-1}G_0(s) \Rightarrow W(s)\Delta(s) = I - \frac{G_0(s)}{G(s)}$$

With the proposed transfer functions, we find

$$1 - \frac{G_0(s)}{G(s)} = 1 - \frac{s+2+\alpha}{s+2} = -\frac{\alpha}{s+2}$$

Hence, with $\Delta(s) = \alpha$, we have $W(s) = -1/(s+2)$.

(d) Identifying $d_{\max} = \alpha_{\max}$, our stability condition reads

$$\|(I + G_0(s))^{-1}W(s)\|_\infty \leq \frac{1}{\alpha_{\max}}$$

Note that

$$(1 + G_0(s))^{-1} W(s) = -\frac{s+1}{(s+2)^2}$$

Using the norm computation from (a) we find the stability condition

$$\frac{\sqrt{3}}{6} \leq \frac{1}{\alpha_{\max}}$$

i.e. that

$$\alpha_{\max} \leq \frac{6}{\sqrt{3}} = \sqrt{12}$$

(e) When $G(s)$ is closed under negative unity feedback $U(s) = -Y(s)$, we find the characteristic polynomial

$$\lambda(s) = (s+2) + (s+1)(s+2+\alpha) = s^2 + (4+\alpha)s + (4+\alpha)$$

The closed-loop system is stable if all coefficients are non-negative, *i.e.* if $\alpha > -4$. Hence, a linear analysis gives that $\alpha_{\max} \leq 4$, while the small gain theorem yields $\alpha_{\max} \leq \sqrt{12}$. Since $\sqrt{12} < \sqrt{16} = 4$, the small gain theorem is more conservative.