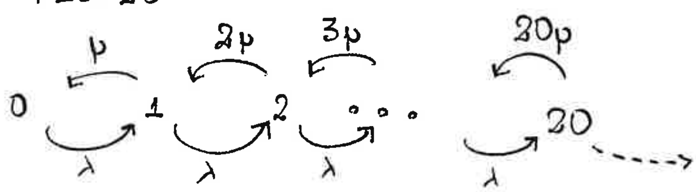
EXAM MAY 20, 2014

1. $m = 20$ servers

$$\left.\begin{aligned} \lambda &= 1 \text{ min}^{-1} \\ \mu &= \frac{1}{15} \text{ min}^{-1} \end{aligned}\right\} \rho = \frac{\lambda}{\mu} = 15$$

a. M/M/20/20



$$P_{block} = P_{20} = E_{20}(15) \cong 0{,}0456$$

$$\lambda_{eff} = \lambda \cdot [1 - E_{20}(15)] = 0{,}954 \text{ calls/min} = 57{,}24 \text{ calls/h}$$

b. Utilization: $P_{busy} = \frac{\rho_{eff}}{m} = 0{,}75 \cdot [1 - 0{,}0456] \cong 0{,}72$

c. $T_{BLOCK} \sim \exp(20\mu) \longrightarrow \overline{T}_{BLOCK} = \frac{1}{20\mu} = 0{,}75 \text{ min}$

$$\frac{\overline{T}_{BLOCK}}{\overline{T}_{BLOCK} + \overline{T}_{ACCEPT}} = E_{20}(15) \longrightarrow \overline{T}_{ACCEPT} \cong 15{,}7 \text{ min}$$

d. M/M/20

$$P_{WAIT} = \frac{m E_m(\rho)}{m - \rho[1 - E_m(\rho)]} \cong 0{,}16$$

$$\overline{W} = \frac{1}{m\mu - \lambda} P_{WAIT} = 0{,}48 \quad , \quad F_W(t) = 1 - D_m(\rho)\, e^{-\mu(m-\rho)t} \cong 0{,}0589$$

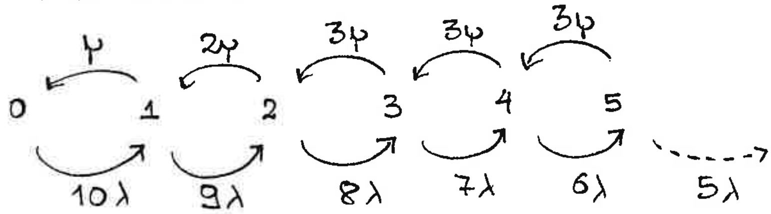
$\lambda_{eff} = \lambda$ (no loss and stable)

(2)

$\lambda = 5h^{-1}$

$\mu = \frac{1}{6} \min^{-1} = 10h^{-1}$

a. M/M/3/5/10



b. Local balance equations

$$\left.\begin{aligned}
10\lambda P_0 &= \mu P_1 \\
9\lambda P_1 &= 2\mu P_2 \\
8\lambda P_2 &= 3\mu P_3 \\
7\lambda P_3 &= 3\mu P_4 \\
6\lambda P_4 &= 3\mu P_5 \\
\sum_{i=0}^{5} P_i &= 1
\end{aligned}\right\} \longrightarrow \left\{\begin{aligned}
P_0 &= 0{,}015 \\
P_1 &= 0{,}075 \\
P_2 &= 0{,}169 \\
P_3 &= 0{,}225 \\
P_4 &= 0{,}2625 \\
P_5 &= 0{,}2625
\end{aligned}\right.$$

$P(\text{buffer full}) = P_5$

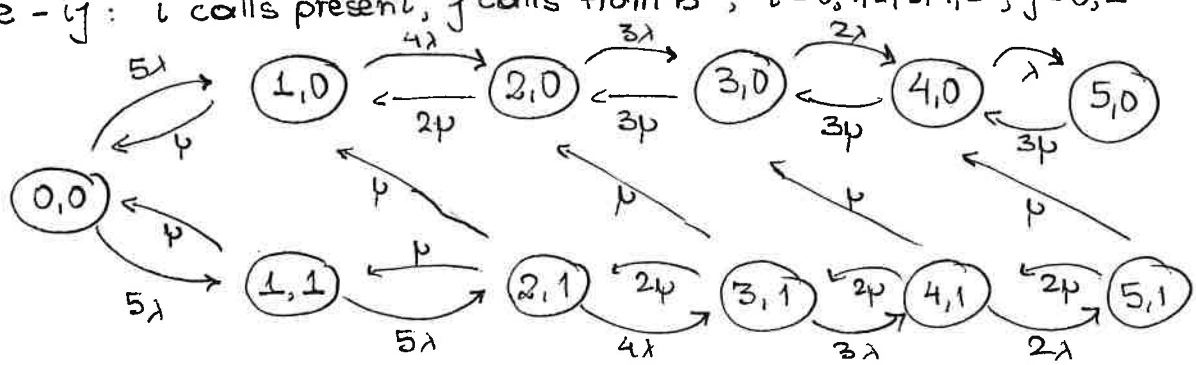
c. $\bar{\lambda}_{block} = 5\lambda \cdot P_5$

d. $\bar{W} = \sum_{i=0}^{5} \bar{W}_i \cdot Pr(\text{arrival sees state } i)$

$\bar{W}_0 = 0, \ \bar{W}_1 = 0, \ \bar{W}_2 = 0, \ \bar{W}_3 = \frac{1}{3\mu}, \ \bar{W}_4 = \frac{2}{3\mu}$

$Pr(\text{arrival sees state } i) = \frac{\lambda_i P_i}{\sum_{k=0}^{4} \lambda_k P_k}, \ i = 0, 1, 2, 3, 4$

e. State - ij : "i calls present, j calls from B", $i = 0,1,2,3,4,5$, $j = 0,1$



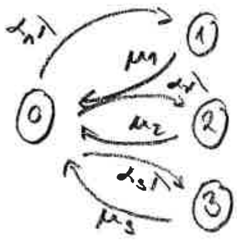
3

$\lambda = 2$

X: $S^*(s) = \frac{1}{s+2} + \frac{1}{s+4} + \frac{2}{s+8} = \frac{1}{2} \cdot \frac{2}{s+2} + \frac{1}{4} \cdot \frac{4}{s+4} + \frac{1}{4} \cdot \frac{8}{s+8}$

$\Rightarrow$ Hyper-Exp: $\alpha_1 = 0.5 \quad \mu_1 = 2$
$\alpha_2 = 0.25 \quad \mu_2 = 4$
$\alpha_3 = 0.25 \quad \mu_3 = 8$

a) $M/H_3/1/1$



$\Rightarrow$
$$\left.\begin{aligned} p_0 &= 2p_1 \\ \frac{p_0}{2} &= 4p_2 \\ \frac{p_0}{2} &= 8p_3 \end{aligned}\right\} \quad p_1 = \frac{p_0}{2}, \quad p_2 = \frac{p_0}{8}, \quad p_3 = \frac{p_0}{16}$$

$p_0\left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{16}\right) = 1$

$p_0 \frac{16+8+2+1}{16} = 1$

$p_0 = \frac{16}{27}, \quad p_1 = \frac{8}{27}$

$p_2 = \frac{2}{27}, \quad p_3 = \frac{1}{27}$

b) Utilization $= 1 - p_0 = \frac{11}{27}$

$P(\text{drop}) = P(\text{server busy}) = 1 - p_0 = \underline{\underline{\frac{11}{27}}}$

c) Infinite buffer capacity $\Rightarrow$ no losses

$\bar{x} = \alpha_1 \cdot \frac{1}{\mu_1} + \alpha_2 \cdot \frac{1}{\mu_2} + \alpha_3 \cdot \frac{1}{\mu_3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{32} = \frac{8+2+1}{32} = \frac{11}{32}$

Utilization $= \rho = \lambda x = 2 \cdot \frac{11}{32} = \underline{\underline{\frac{11}{16}}} > \frac{11}{27}$, since all jobs are accepted.

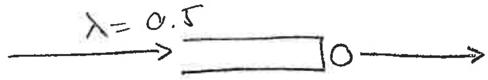
d) $\underline{\underline{\bar{W}}} = \frac{\lambda \overline{x^2}}{2(1-\rho)} =$

$= \frac{\cancel{2} \cdot \frac{37}{128}}{\cancel{2} \cdot \frac{15}{16}} = \frac{37 \cdot \cancel{16}}{\cancel{128}_{8} \cdot 5} = \underline{\underline{0{,}925}}$

$\overline{x^2} = \cancel{2} \cdot \left(\frac{1}{\cancel{2}} \cdot \frac{1}{2^2} + \frac{1}{\cancel{4}_{2}} \cdot \frac{1}{4^2} + \frac{1}{\cancel{4}_{2}} \cdot \frac{1}{8^2}\right) =$

$= \frac{32 + 4 + 1}{128} = \frac{37}{128}$

(4)



Class 1: α $\quad x_1 = \frac{1}{b}$ (Deterministic) $\qquad \alpha = 0.75$

class 2: $1-\alpha$ $\quad x_2 \sim$ Erlang-c, $\quad \bar{x}_2 = \frac{c}{b}$ $\qquad b = 1$, $c = 2$

$\Rightarrow$ Erlang-2, each stage $Exp(b) = Exp(1)$

a) $E[X_1] = \frac{1}{b} = 1$, $\quad E[X_1^2] = \left(\frac{1}{b}\right)^2 = 1$, $\quad V[X_1] = 0$ (Deterministic!)

$E[X_2] = \frac{c}{b} = 2$, $\quad V[X_2] = V[X_{21}] + V[X_{22}] = 2 \cdot \frac{1}{b^2} = 2$, $\quad E[X_2^2] = V[X] + E[X]^2 = 6$

$$E[X] = \alpha E[X_1] + (1-\alpha) E[X_2] = \frac{3}{4} + \frac{1}{4} \cdot 2 = \frac{5}{4}$$

$$E[X^2] = \alpha E[X_1^2] + (1-\alpha) E[X_2^2] = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 6 = \frac{9}{4}$$

$$V[X] = E[X^2] - E[X]^2 = \frac{9}{4} - \frac{25}{16} = \frac{36-25}{16} = \frac{11}{16}$$

b) $$W = \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\frac{1}{2} \cdot \frac{9}{4}}{2\left(1 - \frac{1}{2} \cdot \frac{5}{4}\right)} = \frac{\frac{9}{8}}{2 \cdot \frac{3}{8}} = \frac{3}{2}$$

$$T = W + \bar{x} = \frac{3}{2} + \frac{5}{4} = \frac{11}{4} \approx 2.75$$

c) Preemptive priority:

Class 1: M/D/1

$$W = \frac{\rho_1 x}{1-\rho_1} = \frac{\frac{3}{8} \cdot 1}{\frac{5}{8}} = \frac{3}{5}$$

$$T_1 = W + x = \frac{8}{5}$$

$\rho_1 = \alpha \cdot \lambda \cdot E[X_1] = \frac{3}{8}$

$\rho_2 = (1-\alpha)\lambda E[X_2] = \frac{2}{8}$

$R_2 = \frac{1}{4}\left[\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 6\right] = \frac{9}{16}$

Class 2:

$$T_2 = \frac{(1-\rho_1-\rho_2) \cdot E[X_2] + R_2}{(1-\rho_1)(1-\rho_1-\rho_2)} = \frac{\frac{3}{4} + \frac{9}{16}}{\frac{5}{8} \cdot \frac{3}{8}} = \frac{21}{16} \cdot \frac{64}{15} = \frac{84}{15} = \frac{28}{5}$$

$$T = \alpha T_1 + (1-\alpha) T_2 = \frac{3}{4} \cdot \frac{11}{4} + \frac{1}{4} \cdot \frac{84}{15} \approx 3.46$$

d) The customer under service is surely high priority $\Rightarrow$ than ~~x~~ $2 \cdot x_1 = 2$ time units

$\Rightarrow$ The waiting time is surely less

5

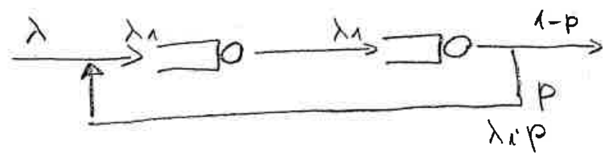
a) $M/G/1$ with vacation.

$\Rightarrow$ All idle time is vacation time

$P(\text{receptionist at his desk}) = \text{utilization} = \lambda \bar{x} = \frac{10}{60} \cdot 5 = \frac{5}{6}$

Time in the cafe per hour: $(1-\rho) \cdot 60\text{min} = 10 \text{ min.}$

b)



$\lambda = 1$

$\mu_1 = \mu_2 = 3$

$p = 0.5$

$\lambda_1 = \lambda + \lambda_1 p$

$\lambda_1 = \frac{\lambda}{1-p} = 2\lambda = 2$

$N_1 = N_2 = \frac{\rho}{1-\rho} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2 \Rightarrow T = \frac{N_1 + N_2}{\lambda} = 4$ time unit.

5c. 2 processors, P.P. (λ).

Interarrival time to each of the processors = sum of two exponential (λ) $\longrightarrow$ $\text{Erlang}_2(\lambda)$

For n processors $\longrightarrow$ $\text{Erlang}_n(\lambda)$

5d. $\lambda = 5 \text{ min}^{-1}$

$\mu = \frac{1}{30} \text{sec}^{-1} = 2 \text{min}^{-1}$

Common queue: we need to dimension an $M/M/m$ system: we need the minimum m, such that $\bar{W} < 15$ sec:

$$\frac{1}{m\mu - \lambda} \cdot \frac{m E_m(\rho)}{m - \rho(1 - E_m(\rho))} < 15$$