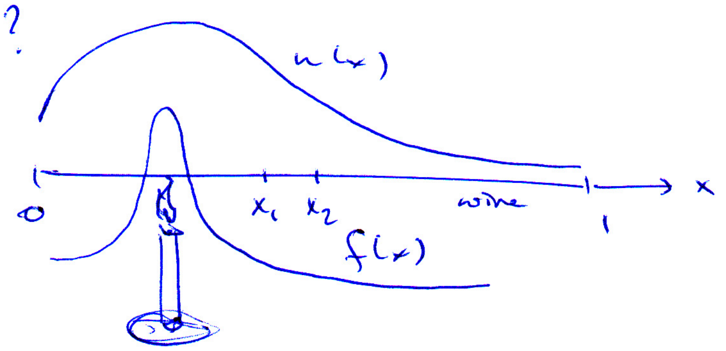
Lecture 1

Heat conduction in a thin heat-conducting wire in $[0,1]$, heated by heat source $f(x)$.

What is the stationary distribution of the temperature $u(x)$?



$q(x)$ - heat flux in the direction of positive x-axis

Energy conservation: for arbitrary sub-interval $(x_1,x_2) \subset (0,1)$: net heat flux through end points = produced heat in (x_1,x_2) per unit time.

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} f(x)\,dx$$

Fundamental Theorem of Calculus: $q(x_2) - q(x_1) = \int_{x_1}^{x_2} q'(x)\,dx$

$$\Rightarrow \int_{x_1}^{x_2} q'(x)\,dx = \int_{x_1}^{x_2} f(x)\,dx$$

(x_1,x_2) arbitrary + assuming $q'(x), f(x)$ cont.

$$\Rightarrow \quad q'(x) = f(x) \quad 0 < x < 1 \quad (\text{Differential eqn.})$$

Constitutive relation: Fourier's law: $q(x) = -a(x)u'(x)$

(Heat flows from warm to cold, proportional to $a(x) > 0$ (coeff. of heat conduct.)

$\Rightarrow$ Stationary heat eqn. $-(a(x)u'(x))' = f(x) \quad 0 < x < 1$

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To define $u(x)$ uniquely: boundary conditions

Ex. Homogeneous Dirichlet b.c. $u(0)=u(1)=0$
(temp. zero at endpoints)

$\Rightarrow$ Two-point boundary value problem

$$\begin{cases} -(au')' = f & \text{in } (0,1) \\ u(0)=u(1)=0 \end{cases}$$

Ex. Homogeneous Neumann b.c. $q(0)=-a(0)u'(0)=0$
(insulating wire at $x=0$)

Non-homogeneous Dirichlet/Neumann b.c.
$u(0)=u_0$, $q(0)=g_0$, ... (prescribed temp/flux)

Robin boundary cond. at $x=1$:

$$a(1)u'(1)+\gamma(u(1)-u_1)=g_1$$

$\gamma \geq 0$ given boundary heat conductivity

$\gamma=0 \Rightarrow$ Neuman; $-a(1)u'(1)=-g_1$

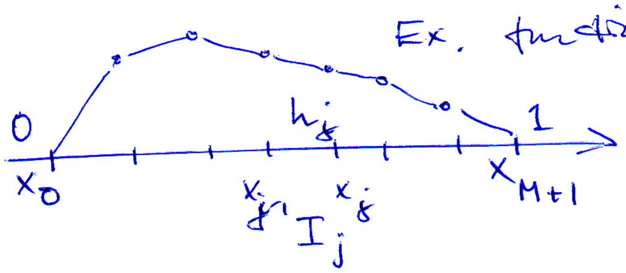
$\gamma=\infty \Rightarrow$ Dirichlet; $u(1)=u_1$

$g_1=0 \Rightarrow$ Heat flux $-a(1)u'(1)$ prop. to temp. difference $u(1)-u_1$
(u_1 - outside temperature)

Find numerical approx. of D.E.

$$(*) \quad \begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

<u>Piecewise linear approximation</u>: on mesh τ_h



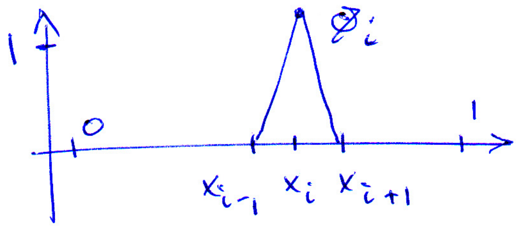
Divide $[0,1]$ into subintervals $I_j = (x_{j-1}, x_j)$

of length $h_j = x_j - x_{j-1}$; τ_h ; $0 = x_0 < x_1 < \dots < x_{M+1} = 1$

$V_h = \{$ set of all cont. p.w. lin. functions on τ_h that are zero at $x = 0$ and $x = 1$. $\}$

V_h finite dimensional vector space of dimension M (M degrees of freedom : nodes)

Basis of V_h : hat functions $\{\varphi_j\}_{j=1}^{M}$



$$\varphi_i(x) = \begin{cases} 0 & x \notin [x_{i-1}, x_{i+1}] \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & x \in [x_i, x_{i+1}] \end{cases}$$

Any function $v(x)$ in V_h can be written

$$v(x) = \sum_{j=1}^{M} v(x_j)\, \varphi_j(x) = \sum_{j=1}^{M} \xi_j\, \varphi_j(x)$$

Construct approx. solution $U(x) = \sum_{j=1}^{M} \xi_j \phi_j(x) \in V_h$

<u>Residual</u> $R(U) = -U'' - f$

(How well $U(x)$ satisfies (*) ; exact sol. $u(x)$: $R(u)=0$)

<u>Galerkin's method</u>: Find $U \in V_h$: $(R(U), v) = 0 \quad \forall v \in V_h$

(Choose $U \in V_h$ s.t. $R(U)$ orthogonal to all $v \in V_h$)

L_2-scalar product in $(0,1)$: $(f,g) = \int_0^1 f(x) g(x)\,dx$

$(R(U), v) = \int_0^1 R(U) v\,dx = \int_0^1 (-U'' - f) v\,dx = 0 \Leftrightarrow -\int_0^1 U'' v\,dx = \int_0^1 fv\,dx$

<u>Weak form / Variational formulation</u>:

$-\int_0^1 u'' v\,dx = \int_0^1 u'v'\,dx - u'(1)v(1) + u'(0)v(0) = \int_0^1 u'v'\,dx$
$= \int_0^1 fv\,dx$

(Since $v \in V_h \Rightarrow v(0) = v(1) = 0$; use part. integration)

<u>Galerkin FEM</u>: Find $U \in V_h$ such that

(G) $\int_0^1 U'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V_h$

<u>Weak form of DE (*)</u>: Find $u \in V$ such that

(W) $\int_0^1 u'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V$

$V = \{$ All functions $v(x)$ ~~such that~~ $v(0) = v(1) = 0$ & integrals are well defined $\}$

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Galerkin orthogonality property: (W) − (G)

$$\int_0^1 (u-U)' v' \, dx = 0 \quad \forall v \in V_h \qquad \text{(G.O.)}$$

(Since all functions in V_h are also in V: $V_h \subset V$)

Norms: measuring the size of a function
(compare absolute value for a number)

L_2-norm: $\|f\| = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}$

Energy-norm: $\|v\|_E = \left(\int_0^1 |v'(x)|^2 dx\right)^{1/2}$

How large is the error $u-U$ in $\|\cdot\|_E$?

$$\|(u-U)'\|^2 = \int_0^1 (u-U)'(u-U)' dx = \left[\pm \int_0^1 v' dx\right]$$

$$= \int_0^1 (u-U)'(u-v)' dx + \int_0^1 \underbrace{(u-U)'(v-U)'}_{\to 0 \text{ (G.O.)}} dx \qquad (v-U) \in V_h$$

$$= \int_0^1 (u-U)'(u-v)' dx \le \|(u-U)'\| \, \|(u-v)'\|$$

[Cauchy-Schwarz ineq. $\int_0^1 fg \, dx \le \|f\| \|g\|$]

$\Rightarrow \quad \|(u-U)'\| \le \|(u-v)'\| \quad \forall v \in V_h$

$\|u-U\|_E \le \|u-v\|_E \quad \forall v \in V_h$

Galerkin approx. optimal in energy norm!

L1

The discrete system of equations

$U(x) = \sum_{j=1}^{n} \xi_j \phi_j(x)$ is determined by $\{\xi_j\}_{j=1}^{n}$

Determine $\{\xi_j\}_{j=1}^{n}$ from (G) ?

(G) $\int_0^1 U'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V_h$

$\Rightarrow \int_0^1 \left(\sum_{j=1}^{n} \xi_j \phi_j(x)\right)' \phi_i'\,dx = \int_0^1 f \phi_i\,dx, \quad i = 1, \dots, M$

($\{\phi_i\}_{i=1}^{n}$ basis of V_h $\Rightarrow$ suff. to check for basis)

$\Rightarrow \sum_{j=1}^{n} \xi_j \int_0^1 \phi_j' \phi_i'\,dx = \int_0^1 f \phi_i\,dx \qquad i = 1, \dots, M$

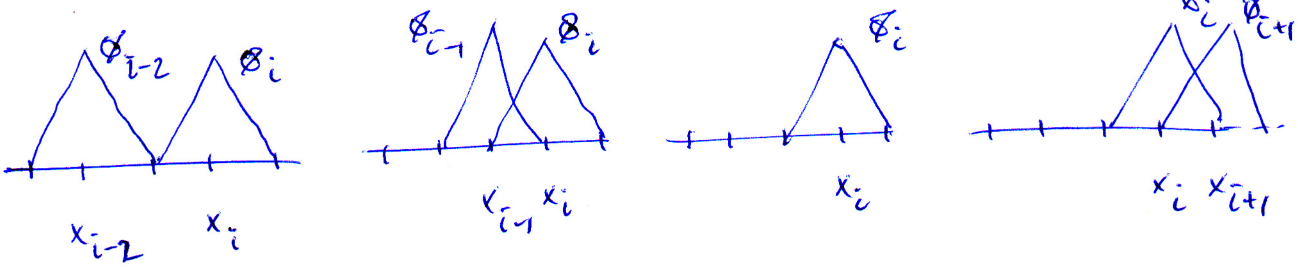
Corresponds to linear system of equations: $A\xi = b$

with matrix $A = (a_{ij})$; vectors $b = (b_i)$, $\xi = (\xi_j)$

$$a_{ij} = \int_0^1 \phi_j' \phi_i'\,dx, \quad b_i = \int_0^1 f \phi_i\,dx$$

A is <u>sparse</u> (most entries are zero) since

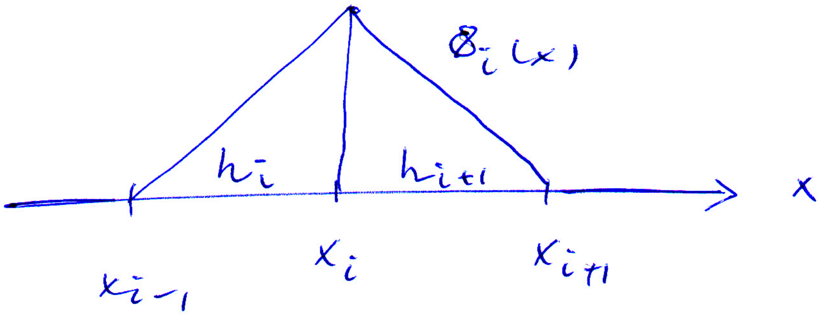
$a_{ij} = 0$ unless $i = j-1, j, j+1$:



$\phi_{i-2} \times \phi_i = 0$

A is symmetric: $a_{ij} = \int_0^1 \phi_j' \phi_i' \, dx = \int_0^1 \phi_i' \phi_j' \, dx = a_{ji}$

$$\phi_i(x) = \begin{cases} (x - x_{i-1})/h_i & x_{i-1} \le x \le x_i \\ (x_{i+1} - x)/h_{i+1} & x_i \le x \le x_{i+1} \end{cases}$$



$$\phi_i'(x) = \begin{cases} \frac{1}{h_i} & x_{i-1} \le x \le x_i \\ -\frac{1}{h_{i+1}} & x_i \le x \le x_{i+1} \end{cases}$$

$$a_{ii} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_i}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{-1}{h_{i+1}}\right)^2 dx = \frac{1}{h_i} + \frac{1}{h_{i+1}}$$

$$a_{i\,i+1} = \int_{x_i}^{x_{i+1}} \frac{-1}{h_{i+1}} \frac{1}{h_{i+1}} \, dx = -\frac{1}{h_{i+1}} = a_{i+1\,i}$$

$$a_{i-1\,i} = \int_{x_{i-1}}^{x_i} \frac{1}{h_i} \frac{-1}{h_i} \, dx = -\frac{1}{h_i} = a_{i\,i-1}$$

$$b_i = \int_{x_{i-1}}^{x_i} f(x) \frac{x - x_{i-1}}{h_i} \, dx + \int_{x_i}^{x_{i+1}} f(x) \frac{x_{i+1} - x}{h_{i+1}} \, dx$$

Further reading

CDE: Ch 1-4 : Background

CDE: Ch 6, 8.1 : Galerkin FEM