Lecture 2

Poisson's equation in $\mathbb{R}^2$ with homogeneous Dirichlet boundary conditions :

$$(D) \quad \begin{cases} -\Delta u(x) = f(x) & x \in \Omega \\ u(x) = 0 & x \in \Gamma \end{cases}$$

Ω bounded domain in $\mathbb{R}^2$ with boundary Γ.

$$\left[\begin{array}{l} -\Delta u = -\frac{\partial^2 u}{\partial x_1} - \frac{\partial^2 u}{\partial x_2}, \quad \text{with } x = (x_1, x_2) \\ \text{"Laplacian of } u(x)\text{"} \end{array} \right]$$

Variational (Weak) formulation :

$$\boxed{\begin{array}{l} \text{Find } u \in V \text{ such that} \\ (W) \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in V \end{array}}$$

where $(w, v) = \int_\Omega w v \, dx$, $(\nabla w, \nabla v) = \int_\Omega \nabla w \cdot \nabla v \, dx$

and $V = \left\{ v : \int_\Omega (|\nabla v|^2 + v^2) dx < \infty \text{ and } v = 0 \text{ on } \Gamma \right\}$

$L_2 = \left\{ v : \int_\Omega v^2 dx < \infty \right\}$

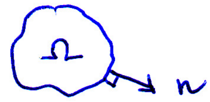
If $u, v \in V$ then $(\nabla u, \nabla v)$ is well defined

If $v \in V$ and $f \in L_2$ then (f, v) — " —

Multiply $-\Delta u = f$ by $v \in V$ and use Green's formula (partial integration in $\mathbb{R}^d$, $d>1$)

$$\int_\Omega f v \, dx = -\int_\Omega \Delta u \, v \, dx = -\int_\Gamma \partial_n u \, v \, ds + \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx$$

(since $v \in V \Rightarrow v = 0$ on Γ)

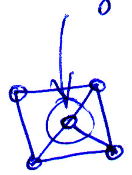


[Normal derivative $\partial_n u = \nabla u \cdot n = \frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2$
with $n = (n_1, n_2)$ outward unit normal]

$\Rightarrow$ (D) and (W) have the same solution
(if f continuous, see Ch. 21)

A triangular mesh of Ω (with a polygonal Γ) $\mathcal{T}_h = \{K\}$ is a sub-division of Ω into a non-overlapping set of triangles/elements/cells K, constructed such that no vertex of one triangle lies on the edge of another triangle:

no "hanging nodes"



$\mathcal{N}_h = \{N\}$ nodes/vertices of $\mathcal{T}_h$

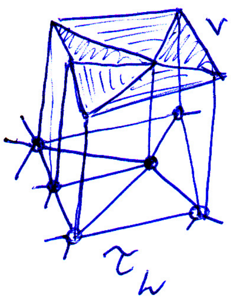
$\mathcal{S}_h = \{S\}$ edges of $\mathcal{T}_h$

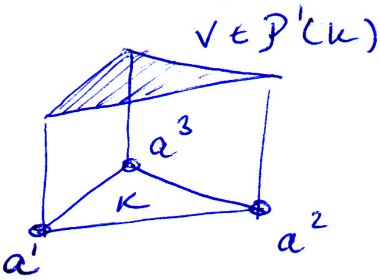
The <u>diameter</u> h_K of a triangle K = longest side of K

<u>Mesh function</u> $h(x)$ of mesh τ_h = p.w. constant function $h(x) = h_K$ for $x \in K$

$$V_h = \{ v : v \text{ continuous on } \Omega, \ v|_K \in P^1(K) \text{ for } K \in \tau_h \}$$

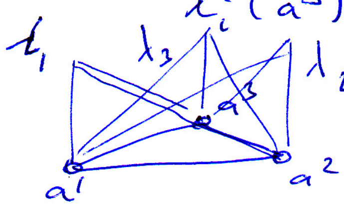
$$P^1(K) = \{ \text{set of linear functions } v(x) \text{ on } K : v(x) = c_0 + c_1 x_1 + c_2 x_2 \}$$

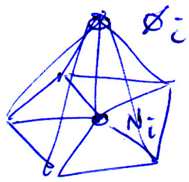




Element basis functions for element K with nodes $\{a^1, a^2, a^3\}$: Element nodal basis functions $\lambda_i \in P^1(K)$ $i = 1,2,3$:

$$\lambda_i(a^j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$





Global basis for V_h : "tent functions"

$$v \in V_h \Rightarrow v(x) = \sum_{i=1}^{M} v(N_i)\, \phi_i(x) \ ; \quad \phi_j(N_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Variational formulation: find $u \in V$:

(V) $\quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in V$

$V = \{ v : \int_\Omega (|\nabla v|^2 + v^2)\, dx < \infty \ \& \ v = 0 \text{ on } \Gamma \}$

$V_h = \{ v \in V : v \text{ p.w. conf. on } \mathcal{T}_h \}$

Degrees of freedom of $v \in V_h$: $\mathcal{N}_h = \{N\}$

set of internal nodes of $\mathcal{T}_h$

(homogeneous Dirichlet b.c.)

Let $\{N_1, \dots, N_M\}$ be an enumeration of $\mathcal{N}_h$

Let $\{\varphi_1, \dots, \varphi_M\}$ be corresponding nodal basis for V_h

Galerkin FEM: Find $U \in V_h$ such that

(G) $\quad (\nabla U, \nabla v) = (f, v) \quad \forall v \in V_h$

$V_h \subset V \Rightarrow$ Galerkin orthogonality: (V)-(G)

$(\nabla u - \nabla U, \nabla v) = 0 \quad \forall v \in V_h$

Discrete system of equations:

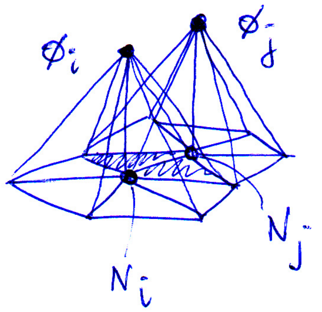
$$U \in V_h \Rightarrow U = \sum_{j=1}^{M} \xi_j \phi_j \quad , \quad \xi_j = U(N_j)$$

(G) & choosing $v = \phi_i \Rightarrow$

$$\sum_{j=1}^{M} (\nabla\varphi_j, \nabla\varphi_i)\, \xi_j = (f, \varphi_i) \qquad i = 1, \dots, M$$

Which is the same as $A\xi = b$

$A = (a_{ij})$; $a_{ij} = (\nabla\phi_j, \nabla\phi_i)$ (stiffness matrix)
$b = (b_i)$; $b_i = (f, \phi_i)$ (load vector)



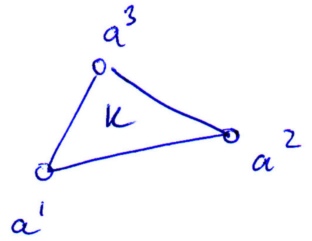
a_{ij} non-zero only if ϕ_i and ϕ_j have common support: that is only if $i = j$ or if N_i and N_j are neighbors.

$a_{ij} = \sum_K a_{ij}^K$; a_{ij}^K ~~element~~ stiffness matrix

contribution for element K

The sum is taken over the common support of ϕ_i and ϕ_j : $\text{supp}(\phi_i) \cap \text{supp}(\phi_j)$

For $K \in \mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j)$; N_i, N_j are nodes of K.



Introduce local numbering of the nodes a^1, a^2, a^3 with corresponding element basis functions $\lambda^1, \lambda^2, \lambda^3$

Elements stiffness matrix (3×3) $A^K_{ij} = (\nabla\lambda_j, \nabla\lambda_i)_K$

with $i,j = 1,2,3$ $\qquad = \int_K \nabla\lambda_j \cdot \nabla\lambda_i \, dx$

local to global index : I^K(local index on K) = global nodal number

$N_i = I^K(a_j)$

<u>Assembly algorithm</u> : to compute global matrix A

For all elements $K \in \mathcal{T}_h$
  Compute element stiffness matrix A^K_{ij}
    For $i,j = 1,2,3$
      Add contribution to global matrix
      $a_{I^K(i) I^K(j)} \mathrel{+}= A^K_{ij}$
    end
end

Assembly of A & $b \Rightarrow A\xi = b$

Solution of system of equation $\rightarrow \xi = (\xi_j)$

Which gives $U(x) = \sum_{j=1} \xi_j \phi_j(x)$

Solution of $A\xi = b$ by :

(1) Direct methods

(2) Iterative methods

For large systems (2) is the only option.

Compare with Puffin where FEM in 2D is implemented for DE in variational form :

$\boxed{\text{www.fenics.org/puffin}}$

Puffin is used for the Computer Projects A + B.

Computation of element stiffness matrix

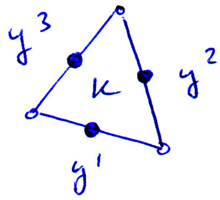
$$A^K_{ij} = \int_K \nabla\lambda_j \cdot \nabla\lambda_i \, dx \qquad i,j = 1,2,3$$

<u>Quadrature</u>: If exact evaluation of integrals is inefficient or impossible

$$\int_K g(x)\,dx \approx \sum_{i=1}^{q} g(y^i)\, w_i$$

q = # quadrature points
y^i = quadrature points (nodes)
w_i = quadrature weights

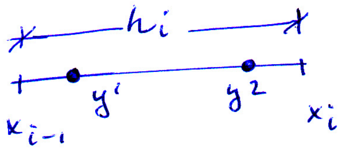
<u>Ex</u>: <u>Midpoint quadrature</u>



$q = 3$, y^i = edge midpoints

$w_i = \frac{|K|}{3}$, $|K|$ = area of K

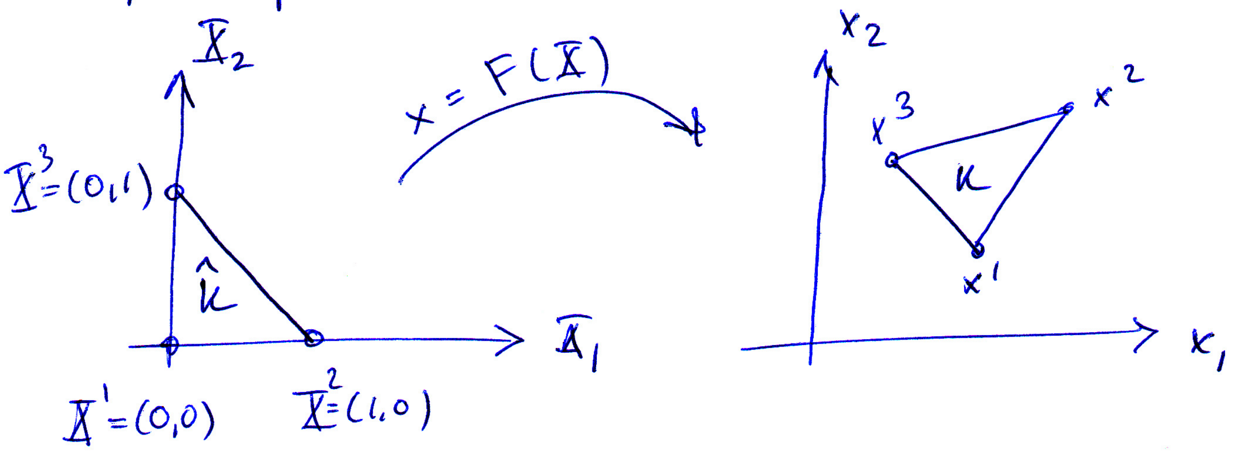
<u>Ex</u>: <u>1D Gauss rule</u>



$q = 2$, $w_i = \frac{h_i}{2}$

$$y^1 = \frac{x_i + x_{i-1}}{2} - \frac{\sqrt{3}}{6} h_i, \quad y^2 = \frac{x_i + x_{i-1}}{2} + \frac{\sqrt{3}}{6} h_i$$

It is common practice to map the triangle K to a reference triangle $\hat{K}$ (reference element), and formulate the quadrature rule in terms of reference coordinates.



<u>Affine mapping</u>: $x = F(\bar{X}) = x^1\,\Phi_1(\bar{X}) + x^2\,\Phi_2(\bar{X}) + x^3\,\Phi_3(\bar{X})$

With Φ_1, Φ_2, Φ_3 nodal basis functions for $\hat{K}$:

$$\Phi_1(\bar{X}) = 1 - \bar{X}_1 - \bar{X}_2 \,, \quad \Phi_2(\bar{X}) = \bar{X}_1 \,, \quad \Phi_3(\bar{X}) = \bar{X}_2$$

$$\nabla_{\bar{X}}\Phi_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \nabla_{\bar{X}}\Phi_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \nabla_{\bar{X}}\Phi_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For the element basis functions of K: $\lambda_1, \lambda_2, \lambda_3$

$$\lambda_1(x) = \Phi_1(\bar{X}) \,, \quad \lambda_2(x) = \Phi_2(\bar{X}) \,, \quad \lambda_3(x) = \Phi_3(\bar{X})$$

The <u>Jacobian $J = F'(\bar{X})$</u> of the mapping:

$$J = F'(\bar{X}) = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} [-1 \ -1] + \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} [1 \ 0] + \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} [0 \ 1]$$

(J constant!)

$$\nabla_x \lambda_i(x) = J^{-1} \nabla_{\bar{X}} \Phi_i(\bar{X}) \quad \left(\text{Solve } J(\nabla_x \lambda_i) = \nabla_{\bar{X}}\Phi_i\right)$$

A function $g(x)$ on K is mapped as

$$g(x) = g(x^1)\,\Phi_1(X) + g(x^2)\,\Phi_2(X) + g(x^3)\,\Phi_3(X)$$

$$\nabla_x g(x) = g(x^1)\,\nabla_X \Phi_1 + g(x^2)\,\nabla_X \Phi_2 + g(x^3)\,\nabla_X \Phi_3$$

For integration: $dx = |J|\, dX$

Element stiffness matrix: $A^K_{ij} = \int_K \nabla \lambda_j \cdot \nabla \lambda_i \, dx$

$$A^K_{ij} = \int_{\hat{K}} \left(J^{-1} \nabla_X \Phi_j(X)\right) \cdot \left(J^{-1} \nabla_X \Phi_i(X)\right) \cdot |J| \, dX$$

Midpoint quadrature $\Rightarrow$

$$A^K_{ij} \approx \sum_{k=1}^{3} \left(J^{-1} \nabla_X \Phi_j(y^k)\right) \cdot \left(J^{-1} \nabla_X \Phi_i(y^k)\right) |J| \, \frac{|\hat{K}|}{3}$$

with $y^1 = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$ $y^2 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ $y^3 = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$

and $|\hat{K}| = \frac{1}{2}$

Further reading:

CDE 15.1 : 2D Poisson eqn.
CDE 13 : Background
CDE 14.1-14.2 : 2D Mesh & p.w. polynomials
CDE 5.5, 14.4 : Quadrature
CDE 7 : optional (not part of the course)