

1.1

(a)

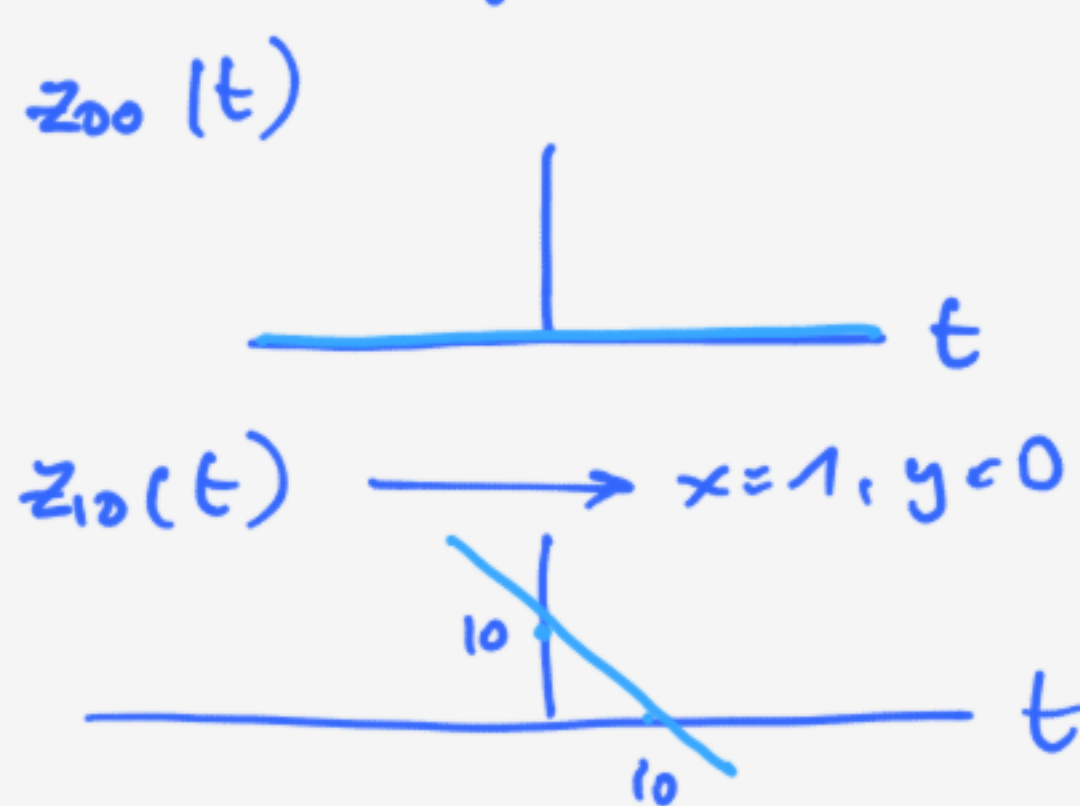
Y and X are independent $\Rightarrow f_{xy}(x,y) = f_x(x)f_y(y)$
 $\Rightarrow E[XY] = E[X]E[Y]$

Additionally $P(X=0) = P(X=1) = P(Y=0) = P(Y=1) = 1/2$

$$Z(t) = (10-t)X + tY$$

Then all possible realizations (deterministic functions of time) :

$z(t)$	for $x=0$	for $x=1$
for $y=0$	0	$10-t$
for $y=1$	t	10



$z_{01}(t) \rightarrow x=0, y=1$



$z_{11}(t)$



$$\begin{aligned} (b) \quad r_z(t_1, t_2) &= E[Z(t_1)Z(t_2)] = E[(10-t_1)X + t_1Y](10-t_2)X + t_2Y) = \\ &= E[(10-t_1)(10-t_2)X^2 + (10-t_1)t_2XY + t_1(10-t_2)YX + t_1t_2Y^2] = \\ &= \left\{ \begin{array}{l} E[\cdot] \text{ is a linear operator} \\ \text{Quantities } t_1 \text{ and } t_2 \text{ are} \\ \text{deterministic} \end{array} \right\} = (10-t_1)(10-t_2)E[X^2] + (10-t_1)t_2E[XY] + \\ &+ t_1(10-t_2)E[YX] + t_1t_2E[Y^2] = \left\{ \begin{array}{l} E[X^2] \rightarrow \left\{ \begin{array}{l} E[g(x)] = \sum_i g(x_i)p(x_i) \text{ (discrete } x) \\ \text{so } E[X^2] = \sum_i x_i^2 p(x_i) = \\ = (1)^2 \cdot \frac{1}{2} + (0)^2 \cdot \frac{1}{2} = \frac{1}{2} \end{array} \right\} \\ E[X^2] = E[Y^2] \Rightarrow \text{same values and same} \\ \text{probabilities.} \\ E[XY] \Rightarrow \left\{ \begin{array}{l} X \text{ and } Y \text{ are} \\ \text{independent} \end{array} \right\} \rightarrow E[XY] = E[X]E[Y] \\ E[X] = \sum_i x_i p(x_i) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2} \end{array} \right. \end{aligned}$$

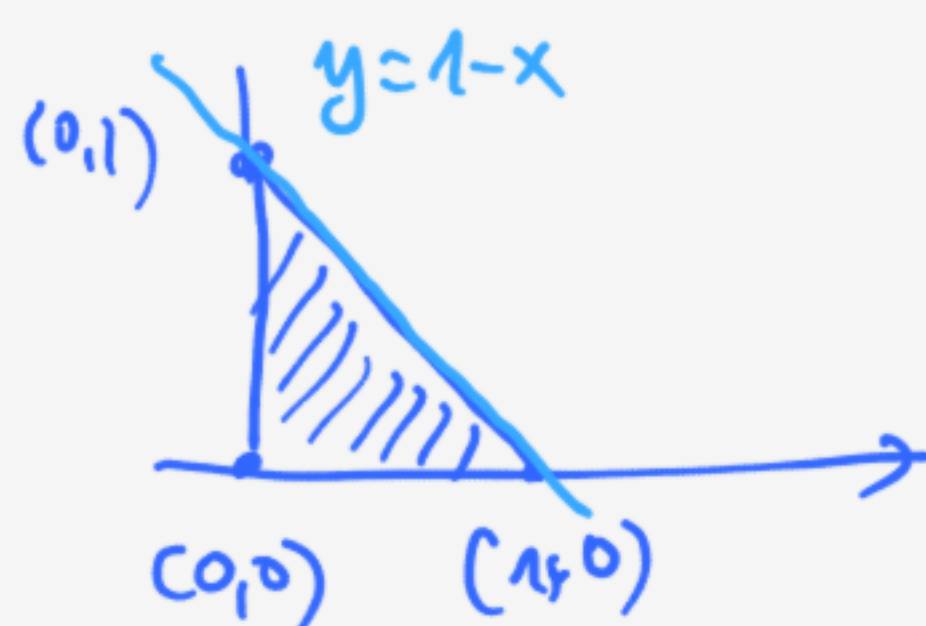
$$= \frac{1}{2} (10-t_1)(10-t_2) + \frac{(10-t_1)t_2}{4} + \frac{t_1(10-t_2)}{4} + \frac{t_1t_2}{2} =$$

$$= \frac{1}{2} (10-t_1 + \frac{t_1}{2})(10-t_2) + \frac{(10-t_1+2t_1)t_2}{4} =$$

$$= \frac{1}{2} (10 - \frac{t_1}{2})(10-t_2) + \frac{(10+t_1)t_2}{4} = \frac{100}{2} - \frac{10}{2}t_2 - \frac{5t_1}{2} + \frac{t_1t_2}{4} +$$

$$+ \frac{10t_2}{4} + \frac{t_1t_2}{4} = 50 - \frac{5t_1}{2} + \left(\frac{10}{4} - \frac{10}{2}\right)t_2 + \frac{2t_1t_2}{4} = 50 - \frac{5t_1}{2} - \frac{5}{2}t_2 + \frac{t_1t_2}{2}$$

1.14



$f_{xy}(x,y)$ is uniform on

$\Downarrow$

$f_{xy}(x,y) = c$ within the triangle. A probability density function needs to fulfill that $\iint f_{xy}(x,y) dx dy = 1$.

And for this case is $\int_0^1 dx \int_0^{1-x} c dy = 1$

$$c \int_0^1 (1-x) dx = c \left(x - \frac{x^2}{2} \right)_0^1 = c \left(1 - \frac{1}{2} \right) = \frac{1}{2} c = 1 \Rightarrow \boxed{c=2}$$

Hence, $f_{xy}(x,y) = 2$ within the triangle and $\emptyset$ outside.

$$\bullet r(X,Y) = E[XY] = \iint xy f_{xy}(x,y) dx dy = \int_0^1 \int_0^{1-x} 2xy dx dy = 2 \int_0^1 x dx \left[\frac{y^2}{2} \right]_0^{1-x} =$$

$$= 2 \int_0^1 x \frac{(1-x)^2}{2} dx = \int_0^1 (x - 2x^2 + x^3) dx = \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1 =$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{6-4+3}{12} = \frac{1}{12} //$$

$$\bullet c(X,Y) = r(X,Y) - m_x m_y = \frac{1}{12} - m_x m_y$$

$$m_x = E[X] = \int x f_x(x) dx$$

$$f_x(x) = \int f_{xy}(x,y) dy = \int_0^{1-x} f_{xy}(x,y) dy = \int_0^{1-x} 2 dy = 2(1-x)$$

depending on which value of x we select, y has a different range of possible values

Additionally we know that $y = 1-x \Rightarrow x = 1-y$, hence:

$$f_y(y) = \int_0^{1-y} f_{xy}(x,y) dx = 2(1-y)$$

Since they have the same distribution for the exact same values. i.e.

$$x \in [0,1] \text{ and } y \in [0,1] \Rightarrow m_x = m_y$$

$$E[X] = \int_0^1 x 2(1-x) dx = 2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 \left(\frac{1}{2} - \frac{1}{3} \right) = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

$$c(X,Y) = \frac{1}{12} - \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{12} - \frac{1}{9} = \frac{3-4}{36} = -\frac{1}{36}$$

$$\bullet \rho(x, y) = \frac{C(x, y)}{\sigma_x \sigma_y} \quad C(x, y) = -1/36.$$

$$\sigma_x = \sqrt{E[(X - \mu_x)^2]}$$

$$E[(X - \mu_x)^2] = E[X^2] - \mu_x^2 = \int x^2 f_x(x) dx - \frac{1}{9} = \int_0^1 x^2 2(x-1) dx - \frac{1}{9} =$$

$$= \left[\frac{2x^4}{4} - \frac{x^3}{3} \right]_0^1 - \frac{1}{9} = \frac{2}{4} - \frac{1}{3} - \frac{1}{9} = \frac{18-12-4}{36} = \frac{2}{36} = \frac{1}{18}$$

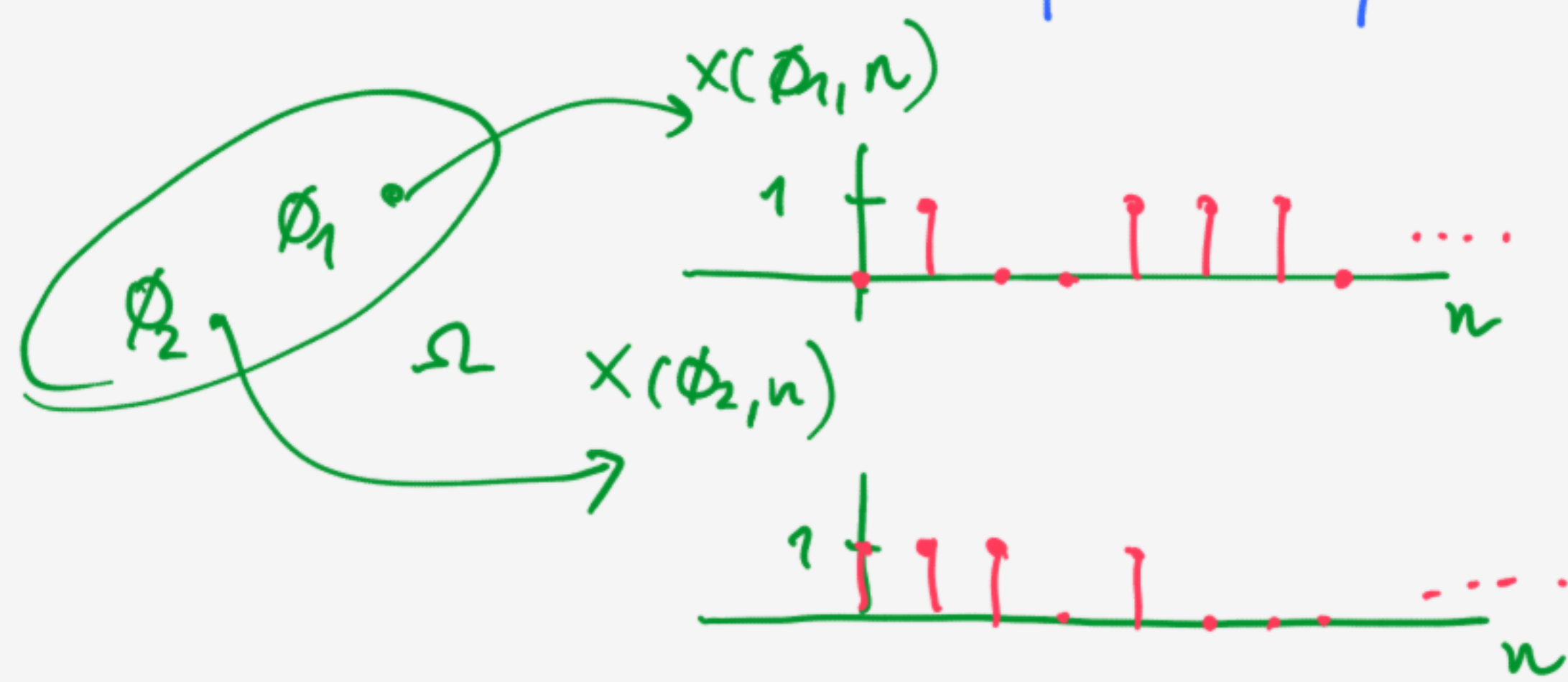
$\sigma_x^2 = \sigma_y^2$ since x and y follow identical p.d.f.'s.

Hence $\sigma_x \sigma_y = \frac{1}{18}$

$$\rho(x, y) = \frac{-1/36}{1/18} = -\frac{18}{36} = -\frac{1}{2}$$

2.2 $X(n)$ is a stationary stochastic process

• each realisation is a sequence of independent 1's and 0's, where 1's and 0's have the same probability



Sequence of independent 1's and 0's means that $X(n_1)$ and $X(n_2)$ are independent

We also know that the process is stationary,

hence $E[X(n_1)] = \mu_x$

$E[X(n_2)] = \mu_x$

$\vdots$
 $E[X(n_m)] = \mu_x \quad \forall n$

Random variables
i.e. $E[X(n_1)X(n_2)] = E[X(n_1)]E[X(n_2)]$

About the random variables we know that they take 0 and 1 with probability $1/2$.

Hence $E[X(n)] = \frac{1}{2} \quad \forall n$ and $E[X(n_1)X(n_2)] = E[X(n_1)]E[X(n_2)]$

which is equivalent to $E[X(n+k)X(n)] = E[X(n+k)]E[X(n)]$

$\uparrow$
 $(k \neq 0)$

(a) $m_y = E[Y(n)] = E\left[\frac{1}{2}(X(n) + X(n-1))\right] \Rightarrow \left. \begin{array}{l} \text{Linear and} \\ \text{constants get} \\ \text{out} \end{array} \right\}$
 $= \frac{1}{2} (E[X(n)] + \underbrace{E[X(n-1)]}_{\substack{\downarrow \\ E[X(n)] \\ \text{(stationary!)}}}) = \frac{2}{2} E[X(n)] = 1/2$

$$\begin{aligned} (b) \quad P_Y &= E[Y(n)^2] = \frac{1}{4} E[(X(n) + X(n-1))^2] = \\ &= \frac{1}{4} (E[X^2(n)] + 2X(n)X(n-1) + E[X(n-1)^2]) = \\ &= E[X(n)^2] \quad (\text{stationarity!}) \end{aligned}$$

$$= \frac{1}{4} (2E[x^2(n)] + 2E[x(n)x(n-1)]) = \frac{1}{2} E[x^2(n)] + \frac{1}{2} E[x(n)] E[x(n-1)] =$$

↑
samples are independent

$$= E[x(n)]$$

$$= \frac{1}{2} E[X^2(n)] + \frac{1}{2} E[X(n)]^2 = \left\{ \begin{array}{l} E[X^2(n)] = \frac{1}{2}(1)^2 + \frac{1}{2}(0) \\ E[X(n)] = 1/2 \end{array} \right\} =$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \left(\frac{1}{4} \right) = \frac{1}{4} + \frac{1}{8} = \frac{2+1}{8} = \frac{3}{8}$$

- variance σ_y^2

$$v_y^2 = p_y - m_y^2 = \frac{3}{8} - \left(\frac{1}{2}\right)^2 = \frac{3}{8} - \frac{1}{4} = \frac{3-2}{8} = \frac{1}{8}$$

- ACF.

A sample (random variable due to fix time in a stochastic process) of $Y(n_1) = \frac{1}{2} (X(n_1) + X(n_1-1))$ is the linear combination of two random variables with specific statistical properties.

For any other sample of the process

$Y(n_2) = \frac{1}{2} (X(n_2) + X(n_2-1))$ the random variable will have the same identical statistical properties as before. (due to $X(n)$ being stationary)

Since this holds $\forall n$ $\chi(n)$ is stationary.

Then $r_y(n_1, n_2)$ will only depend on the sample shift $k = n_1 - n_2$.

$$r_y(k) = E[Y(n+k)Y(n)] = \frac{1}{4} E[(X(n+k) + X(n+k-1))(X(n) + X(n-1))] =$$

$$= \frac{1}{4} E[\underbrace{X(n+k)X(n)}_{r_x(k)} + \underbrace{X(n+k)X(n-1)}_{r_x(k+1)} + \underbrace{X(n+k-1)X(n)}_{r_x(k-1)} + \underbrace{X(n+k-1)X(n-1)}_{r_x(k)}]$$

$r_x(k)$ shift between the two $\rightarrow$ $r_x(k+1)$ $r_x(k-1)$ $r_x(k)$

$$= \frac{1}{4} (2r_x(k) + r_x(k+1) + r_x(k-1)) = \left\{ \begin{array}{l} r_x(k) \left\{ \begin{array}{l} \bullet k=0 \quad E[X^2(n)] = \\ = \frac{1}{2}(1)^2 + \frac{1}{2}(0)^2 = \frac{1}{2} \\ \bullet k \neq 0 \quad E[X(n)X(n+k)] \rightarrow \\ \rightarrow \text{independent} \\ \text{of other} \\ \text{samples} \\ = E[X(n)]E[X(n+k)] = \\ = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \end{array} \right. \end{array} \right.$$

If all arguments of the correlations above are different than 0 we have

$$r_y(k) = \frac{1}{4} \left(2 \cdot \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) = \frac{1}{4}.$$

The only case in which one of the arguments will be 0 is for $k=0$, $k=1$ and $k=-1$

Hence : • $r_y(0) = \frac{1}{4} (2r_x(0) + r_x(1) + r_x(-1)) = \frac{1}{4} \left(2 \cdot \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \right) =$

$$= \frac{1}{4} \left(1 + \frac{1}{2} \right) = \frac{1}{4} + \frac{1}{8} //$$

• $r_y(1) = \frac{1}{4} (2r_x(1) + r_x(0) + r_x(2)) = \frac{1}{4} \left(2 \cdot \frac{1}{4} + \frac{1}{2} + \frac{1}{4} \right) =$

$$= \frac{1}{4} \left(1 + \frac{1}{4} \right) = \frac{1}{4} + \frac{1}{16} //$$

• $r_y(-1) = r_y(1)$ (property of acf. for stationary processes)

$$r_y(k) = \frac{1}{4} + \left\{ \begin{array}{l} \frac{1}{8} \quad \text{for } k=0 \\ \frac{1}{16} \quad \text{for } k=-1 \text{ or } 1 \\ 0 \quad \text{for the rest} \end{array} \right.$$

2.6.

$X(t)$ and $Y(t)$ are independent and wide sense stationary stochastic processes. $\Rightarrow E[X(t)Y(t)] = E[X(t)]E[Y(t)]$
 $\uparrow$
independent

$$\begin{aligned} \Rightarrow E[X(t)] = m_x \quad E[Y(t)] = m_y \quad r_x(t_1, t_2) = r_x(\tau) \quad (\tau = t_1 - t_2) \\ \uparrow \text{wide sense stationary} \quad r_y(t_1, t_2) = r_y(\tau) \end{aligned}$$

$$Z(t) = X(t) + Y(t)$$

$$U(t) = X(t) \cdot Y(t)$$

• when 2 processes are independent
 $\Rightarrow$ all random variables that come from one of them are independent of all the ones that come from the other

- For the processes to be wide sense stationary the mean has to be constant and the a.c.f. shall depend only on the shift.

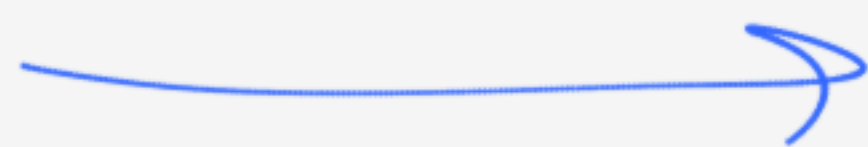
$\rightarrow$ Mean:

$$E[Z(t)] = E[X(t) + Y(t)] \rightarrow \text{linear} = E[X(t)] + E[Y(t)] = m_x + m_y.$$

$$E[U(t)] = E[X(t)Y(t)] = E[X(t)]E[Y(t)] = m_x m_y.$$

$\uparrow$
independent

For both the mean is constant!



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→ auto-correlation function :

* Assume non-stationarity :

$$\begin{aligned} \text{ACF}(z(t)) &= E[Z(t_1)Z(t_2)] = E[(X(t_1) + Y(t_1))(X(t_2) + Y(t_2))] \\ &= \left. \begin{array}{l} \text{the expectation} \\ \text{is linear} \end{array} \right\} = \underbrace{E[X(t_1)X(t_2)]}_{r_x(t_1, t_2) = r_x(\tau) \quad (\tau = t_1 - t_2)} + E[X(t_1)Y(t_2)] + \\ &+ E[X(t_2)Y(t_1)] + E[Y(t_1)Y(t_2)] = r_x(\tau) + r_y(\tau) + \end{aligned}$$

↑
x and y are
w.s. stationary

$$+ E[X(t_2)]E[Y(t_1)] + E[X(t_1)]E[Y(t_2)] = \left. \begin{array}{l} X \text{ and } Y \text{ are} \\ \text{independent} \end{array} \right\}$$

$$= r_x(\tau) + r_y(\tau) + 2m_x m_y \quad \blacksquare$$

↑
w.s. stationarity
X and Y

$$\begin{aligned} \text{ACF}(u(t)) &= E[u(t_1)u(t_2)] = E[X(t_1)Y(t_1)X(t_2)Y(t_2)] = \\ &= \left. \begin{array}{l} X \text{ and } Y \\ \text{are independent} \end{array} \right\} = E[X(t_1)X(t_2)]E[Y(t_1)Y(t_2)] = \\ &= \left. \begin{array}{l} X \text{ and } Y \\ \text{are w.s. stationary} \end{array} \right\} = r_x(\tau)r_y(\tau). \quad \blacksquare \end{aligned}$$

2.11

A(k) : Cannot be an ACF of a wide-sense stationary process since it has points that have the same modulus as A(0) but is not periodic (it ends).

B(k) : B(0) is the maximum value in modulus
it's symmetric with k
It CAN be an ACF.

C(k) : it cannot since $C(0) < C(1)$
 $C(0) < C(2)$

D(k) : it cannot since $D(0) < |D(2)| = |D(-2)|$