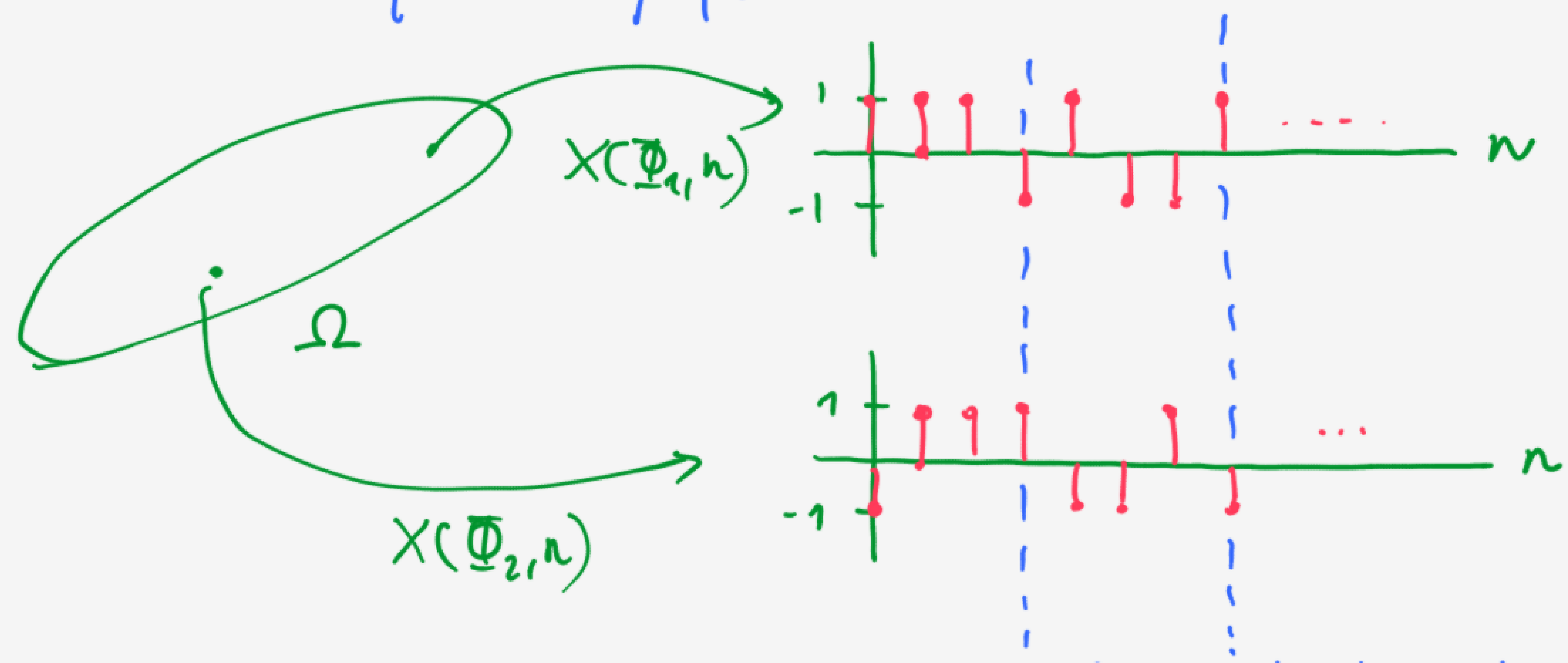
2.4

- $x(n)$ is a stationary stochastic process
- each realisation sequence of independent values $+1, -1$ with -1 with probability p.



$X(n_1)$ random variable $\quad$ $X(n_2)$ random variable

sequence of independent values means $X(n_1)$ and $X(n_2)$ are independent
$\Rightarrow E[X(n_1)X(n_2)] = E[X(n_1)]\,E[X(n_2)]$

We aditionally know that the process is stationary, which in practice will imply that
$E[X(n_1)] = E[X(n_2)] \;\; \forall n_1, n_2.$
and
$$E[X(n_1)X(n_2)] = r_X(\underbrace{n_1-n_2}_{K}) = r_X(k).$$

Since we know the values that $X(n_1), X(n_2) \ldots,$ (all n) can take and with which probability:

- $\rightarrow$ mean $E[X(n)] = \sum_1' x\,p(x) = (-1)\,p + 1(1-p) = 1-2p$
- $\rightarrow$ quadratic mean $E[X^2(n)] = \sum_1' x^2 p(x) = (-1)^2 p + (1)^2(1-p) = 1$

(a) Mean of $Y(n) \rightarrow m_y$

$$m_y = E[Y(n)] = E[X(n)X(n-1)] = \left\{\begin{matrix}\text{independence}\\ \text{of different samples}\end{matrix}\right\} =$$
$$= E[X(n)]\,E[X(n-1)] = \{\text{stationarity}\} = E[X(n)]\,E[X(n)] = (1-2p)^2 \;//$$

(b) $$r_y(k) = E[(\underbrace{X(n+k)\,X(n+k-1)}_{\text{will always be independent of each other.}})(\underbrace{X(n)\,X(n-1)}_{\text{will always be independent of each other}})] =$$

some will be dependent when the time instances are the same.

i.e. $X(n+k)$ will be dependent $\begin{cases} \text{with } X(n) \text{ if } k=0 \\ \text{with } X(n-1) \text{ if } k=-1 \end{cases}$

$X(n+k-1)$ will be dependent $\begin{cases} \text{with } X(n) \text{ if } k=1 \\ \text{with } X(n-1) \text{ if } k=0 \end{cases}$

Hence special cases will be $\left\{ \begin{array}{l} k=0 \\ k=-1 \\ k=+1 \end{array} \right\}$ for all the other values of k the four elements will be independent of each other

So :

- $k \neq 0, |k| \neq 1$:

$$E[X(n+k)X(n+k-1)X(n)X(n-1)] = \{\text{independence}\} =$$

$$= E[X(n+k)]E[X(n+k-1)]E[X(n)]E[X(n-1)] =$$

$$= \{\text{stationaeity}\} = E[X(n)]^4 = (1-2p)^4$$

- $r_Y(0) = E[X(n)X(n-1)X(n)X(n-1)] = \{\text{independence}\}$

$$= E[X(n)^2]E[X(n-1)^2] = \{\text{stationarity}\} = E[X(n)^2]^2 =$$

$$(1)^2 = 1$$

- $r_Y(1) = E[X(n+1)X(n)X(n)X(n-1)] = \{\text{independence}\} =$

$$= E[X(n+1)]E[X^2(n)]E[X(n-1)] = \{\text{stationaeity}\} =$$

$$= E[X(n)]^2 E[X^2(n)] = (1-2p)^2 \cdot 1 = (1-2p)^2$$

- $r_Y(-1) \rightarrow$ symmetry! $r_Y(-1) = r_Y(1)$

$$r_Y(k) = \begin{cases} 1 & k=0 \\ (1-2p)^2 & |k|=1 \\ (1-2p)^4 & \text{rest} \end{cases}$$

$\boxed{2.8}$

X, Y are two independent 0 mean random variables with variances $E[X^2] = \sigma_x^2$, $E[Y^2] = \sigma_y^2$

$$z(n) = X\cos(2\pi\nu n) + Y\sin(2\pi\nu n)$$

• $m_z = E[z(n)] = \left\{\begin{array}{l}\text{linear and}\\ \text{constants are pulled}\\ \text{out}\end{array}\right\} = E[\cancel{X}]^{\to 0}\cos(2\pi\nu n) +$

$E[\cancel{Y}]^{\to 0}\sin(2\pi\nu n) = 0$ //

• $E[z(n_1)z(n_2)] = E[(X\cos(2\pi\nu n_1) + Y\sin(2\pi\nu n_1))(X\cos(2\pi\nu n_2 + Y\sin(2\pi\nu n_2))] = \left\{\begin{array}{l}\text{linear and constants}\\ \text{get pulled out}\end{array}\right\} = E[X^2]\cos(2\pi\nu n_1)\cos(2\pi\nu n_2)$

independent + 0 mean ⇒ 0

$+ E[\cancel{XY}]^{\to \emptyset}\cos(2\pi\nu n_1)\sin(2\pi\nu n_2) + E[\cancel{YX}]^{\to 0}\sin(2\pi\nu n_1)\cos(2\pi\nu n_2) +$

$+ E[Y^2]\sin(2\pi\nu n_1)\sin(2\pi\nu n_2) = \left\{\begin{array}{l}\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))\\ \sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))\end{array}\right\}$

$= \frac{\sigma_x^2}{2}(\cos(2\pi\nu(n_1+n_2)) + \cos(2\pi\nu(\underbrace{n_2-n_1}_{n_2-n_1=k})) + \frac{\sigma_y^2}{2}(\cos(2\pi\nu(\underbrace{n_2-n_1}_{=k}))$

$- \cos(2\pi\nu(n_2+n_1))$

we need to get rid of the terms with n_2+n_1 for the process to be wide sense stationary. We need that :

$$\frac{\sigma_x^2}{2}(\cos(2\pi\nu(n_1+n_2)) - \frac{\sigma_y^2}{2}(\cos(2\pi\nu(n_2+n_1))) = 0$$

which can be achieved if $\sigma_x^2 = \sigma_y^2$. //

And then the ACF :

$r_z(k) = \sigma_x^2(\cos(2\pi\nu k))$ $(\sigma_x^2 = \sigma_y^2)$.

$\boxed{2.10}$ $X(t)$ and $Y(t)$ are independent stationary Gaussian processes with:

$m_X = 1 \quad \sigma_X = 2$

$m_Y = 3 \quad \sigma_Y = 4$

- $X(t)$ are independent $Y(t)$
$\Rightarrow$ all random variables originating of one of them are independent of those of the other.

- A gaussian random variable is completely characterized by its mean and variance.
- Any linear combination of independent Gaussian random variables is a Gaussian random variable.

Calculate the probability of $Z(t) > 0 \Rightarrow Z(t) = 1 \Rightarrow X(t) \geq Y(t)$

If we had any other kind of random variables (not Gaussian) we would do the following:

$$P(Z(t)=1) = \int P(X(t) \geq a, Y(t) \leq a)\, da$$

But since we know that the linear combination of independent Gaussian random variables is gaussian, we know that $A(t) \triangleq X(t) - Y(t)$ is a Gaussian random variable for each t.

Hence we are able to characterize it by obtaining its mean and variance:

$\rightarrow E[A(t)] = E[X(t) - Y(t)] \rightarrow$ linearity $= E[X(t)] - E[Y(t)] =$
$= 1 - 3 = -2.$

$\rightarrow E[A^2(t)] = E[(X(t) - Y(t))^2] = E[X^2(t)] - 2E[X(t)Y(t)]$
$+ E[Y^2(t)] = (\sigma_X^2 + m_X^2) - 2m_X m_Y + (\sigma_Y^2 + m_Y^2) =$
$= (4+1) - 2(1 \cdot 3) + (16+9) = 5 - 6 + 25 = 24$

$\sigma_A^2 = 24 - (-2)^2 = 20.$

Then the random variables that originate from the process $A(t)$ can be characterized as:

$$f_{A(t)}(a) = \frac{1}{\sqrt{2\pi\sigma_a^2}}\, e^{-\frac{(a-m_a)^2}{2\sigma_a^2}}$$

Hence, $P(A(t) \geq 0) = \int_0^{\infty} \frac{1}{\sqrt{2\pi\, 20}} \, e^{-\frac{(a+2)^2}{2 \cdot 20}} \, da.$

This primitive can not be obtained → Leads to the error function / Q function which we know the values using charts.

The error function: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt$

Engineers preffer the Q function:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} \, dt$$

We will have to adapt our integral to "look" more like the Q function:

we introduce the change of variables:

$b = \frac{a+2}{\sqrt{20}}$ $\qquad db = \frac{da}{\sqrt{20}}$

$$P(A \geq 0) = \int_{\frac{2}{\sqrt{20}}}^{\infty} \frac{1}{\sqrt{2\pi \cancel{20}}} \, e^{-\frac{1}{2}b^2} \, \cancel{\sqrt{20}} \, db = \frac{1}{\sqrt{2\pi}} \int_{\frac{2}{\sqrt{20}}}^{\infty} e^{-\frac{1}{2}b^2} \, db =$$

when $a = 0$ $\; b = \frac{2}{\sqrt{20}}$

$$= Q\left(\frac{2}{\sqrt{20}}\right) \approx 0{,}326$$