Appendix A: Cylinderkoordinater

De tre koordinaterna är (r, θ, z), där r är den radiella koordinaten, θ den azimutala (eller den transverella) kordinaten och z den axiella koordinaten. En vektor $\mathbf{u}$ kan delas upp i komponenter enligt

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z \, . \tag{1}$$

De båda enhetsvektorerna $\mathbf{e}_r$ och $\mathbf{e}_\theta$ är funktioner av θ, d v s de vrider sig med θ. Följande samband gäller

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \, , \tag{2}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \, . \tag{3}$$

Gradienten av en skalär

$$\nabla \phi = \mathbf{e}_r \frac{\partial \phi}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \mathbf{e}_z \frac{\partial \phi}{\partial z} \, . \tag{4}$$

Divergensen av en vektor

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \, . \tag{5}$$

Rotationen av en vektor

$$\nabla \times \mathbf{u} = \mathbf{e}_r \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) + \mathbf{e}_\theta \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) + \mathbf{e}_z \left(\frac{1}{r} \frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \tag{6}$$

Laplace av en skalär

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \, . \tag{7}$$

Laplace av en vektor

$$\nabla^2 \mathbf{u} = \mathbf{e}_r \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) + \mathbf{e}_\theta \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right) + \mathbf{e}_z \nabla^2 u_z \tag{8}$$

Inkompressibla Navier-Stokes ekvationer utan gravitationskraft

$$\frac{\partial u_r}{\partial t} + (\mathbf{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right), \tag{9}$$

$$\frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right), \tag{10}$$

$$\frac{\partial u_z}{\partial t} + (\mathbf{u} \cdot \nabla) u_z = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 u_z, \tag{11}$$

där

$$\mathbf{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}. \tag{12}$$

Töjningstensorn

$$e_{rr} = \frac{\partial u_r}{\partial r}, \tag{13}$$

$$e_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \tag{14}$$

$$e_{zz} = \frac{\partial u_z}{\partial z}, \tag{15}$$

$$e_{r\theta} = \frac{r}{2} \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{2r} \frac{\partial u_r}{\partial \theta}, \tag{16}$$

$$e_{\theta z} = \frac{1}{2r} \frac{\partial u_z}{\partial \theta} + \frac{1}{2} \frac{\partial u_\theta}{\partial z}, \tag{17}$$

$$e_{zr} = \frac{1}{2} \frac{\partial u_r}{\partial z} + \frac{1}{2} \frac{\partial u_z}{\partial r}. \tag{18}$$