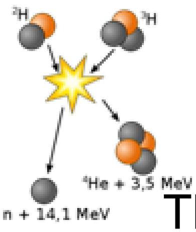
# VEKTORANALYS

Kursvecka 4

# NABLA OPERATOR, INTEGRALSATSER and CARTESIAN TENSORS

(indexräkning)

Kapitel 8-9
Sidor 83-98

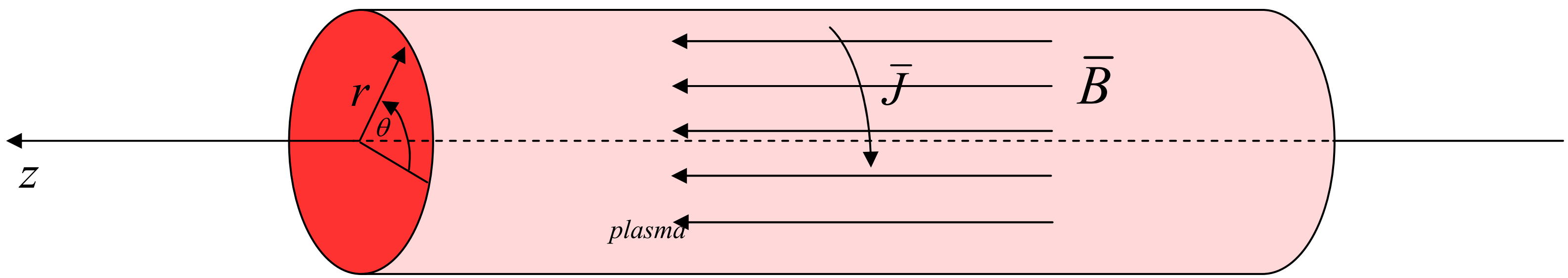



# TARGET PROBLEM

There are millions of particles, strong magnetic fields and electric currents…
How can we describe the behaviour of the plasma?

**Magnetohydrodynamics (MHD)**

(Very) simple example:




When the plasma is in equilibrium, MHD equations are:

$$\begin{cases} grad\ p = \bar{j} \times \bar{B} \\ rot\bar{B} = \mu_0 \bar{j} \end{cases} \Rightarrow \quad grad\ p = \frac{1}{\mu_0}\left(rot\bar{B}\right) \times \bar{B}$$

*And then?*
*How to continue?*

*$p$ is the pressure*
*$\bar{j}$ is the current density*

We need to introduce:

- **Operators**
- **Nabla**

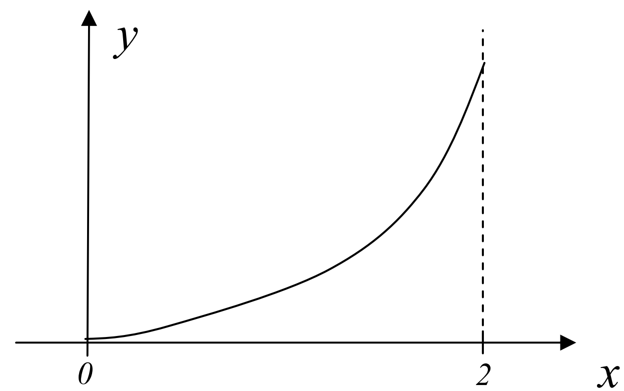
# OPERATOR

What is a function?

A function is a law defined in a domain $X$ that to each element $x$ in $X$ associates one and only one element $y$ in $Y$.

Example: $X=[0,2]$

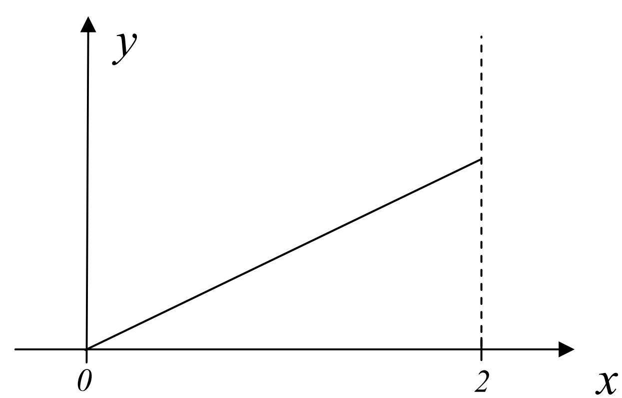
$f(x)=x^2$




The slope of $f(x)$ is its derivative:

$$g(x) = \frac{df(x)}{dx}$$

$g(x)$ is still a function.




So the derivative is **a rule that associates a function to another function**.

The derivative is an example of **operator**

# OPERATOR

DEFINITION

An **operator** $T$ is a law that to each function $f$ in the function class $D_t$ associates a function $T(f)$.

DEFINITION

An operator $T$ is **linear** if ***$T(af+bg)=aT(f)+bT(g)$***, where $f$ and $g$ are functions belonging to $D_t$.

EXAMPLE: $T = \frac{d}{dx}$ *is it linear?* YES

$$T(af+bg) = \frac{d(af+bg)}{dx} = a\frac{df}{dx} + b\frac{dg}{dx} = aT(f) + bT(g)$$

# SUM AND PRODUCT OF OPERATORS

Sum of two operators $(T+U)(f) = T(f) + U(f)$

Product of two operators $(TU)(f) \quad = T(U(f))$

# NABLA

We studied gradient, divergence and curl:

$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$ is common to all three definitions

$$\nabla \equiv \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

*This operator is called* **NABLA**

$$grad\,\phi \equiv \left(\frac{\partial \phi}{\partial x},\frac{\partial \phi}{\partial y},\frac{\partial \phi}{\partial z}\right) \qquad grad\,\phi = \nabla \phi$$

$$div\overline{A} \equiv \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} \qquad div\overline{A} = \nabla \cdot \overline{A}$$

$$rot\,\overline{A} \equiv \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad rot\,\overline{A} = \nabla \times \overline{A}$$

# LAPLACE OPERATOR and something more

- The divergence of the gradient is called laplacian or **Laplace operator**

$\nabla \cdot \nabla \phi = \nabla^2 \phi$ is the laplacian of the scalar field $\phi$. *Sometimes written as:* $\Delta\phi$

In cartesian coordinates: $\nabla^2 = \nabla \cdot \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$

$$\nabla^2 \phi = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)$$

- The nabla can be used to define new operators like: $\overline{A} \cdot \nabla$ or $\overline{A} \times \nabla$

Example: $\overline{A} \cdot \nabla = \left(A_x, A_y, A_z\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}\right)$

so: $$\left(\overline{A} \cdot \nabla\right)\overline{B} = \left(A_x \frac{\partial \overline{B}}{\partial x} + A_y \frac{\partial \overline{B}}{\partial y} + A_z \frac{\partial \overline{B}}{\partial z}\right)$$

Note that: $\left(\overline{A} \cdot \nabla\right)\overline{B} \neq \overline{A}\left(\nabla \cdot \overline{B}\right)$

# VECTOR IDENTITIES

$\phi$ *and* $\psi$ : *scalar fields*

$\overline{A}$ *and* $\overline{B}$ : *vector fields*

$$\nabla\left(\phi\psi\right)=\left(\nabla\phi\right)\psi+\phi\left(\nabla\psi\right) \qquad \textbf{ID1}$$

$$\nabla\cdot\left(\phi\overline{A}\right)=\left(\nabla\phi\right)\cdot\overline{A}+\phi\nabla\cdot\overline{A} \qquad \textbf{ID2}$$

$$\nabla\times\left(\phi\overline{A}\right)=\left(\nabla\phi\right)\times\overline{A}+\phi\nabla\times\overline{A} \qquad \textbf{ID3}$$

$$\nabla\cdot\left(\overline{A}\times\overline{B}\right)=\overline{B}\cdot\left(\nabla\times\overline{A}\right)-\overline{A}\cdot\left(\nabla\times\overline{B}\right) \qquad \textbf{ID4}$$

$$\nabla\times\left(\overline{A}\times\overline{B}\right)=\left(\overline{B}\cdot\nabla\right)\overline{A}-\overline{B}\left(\nabla\cdot\overline{A}\right)-\left(\overline{A}\cdot\nabla\right)\overline{B}+\overline{A}\left(\nabla\cdot\overline{B}\right) \qquad \textbf{ID5}$$

$$\nabla\left(\overline{A}\cdot\overline{B}\right)=\left(\overline{B}\cdot\nabla\right)\overline{A}+\left(\overline{A}\cdot\nabla\right)\overline{B}+\overline{B}\times\left(\nabla\times\overline{A}\right)+\overline{A}\times\left(\nabla\times\overline{B}\right) \qquad \textbf{ID6}$$

$$\nabla\times\left(\nabla\phi\right)=0 \qquad \textbf{ID7}$$

$$\nabla\cdot\left(\nabla\times\overline{A}\right)=0 \qquad \textbf{ID8}$$

$$\nabla\times\left(\nabla\times\overline{A}\right)=\nabla\left(\nabla\cdot\overline{A}\right)-\nabla^2\overline{A} \qquad \textbf{ID9}$$

*See next slides for the proof of some of these identities*

# NABLARÄKNING

Let's consider **ID2**: $\nabla\cdot\left(\phi\overline{A}\right)=\underbrace{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right)}\cdot\left(\phi\overline{A}\right)$

*This seems almost like a vector!*

*Can we simply use the vector algebra rules?* ***NO!***

Nabla contains derivatives and we know that: $\frac{d(fg)}{dx}=\frac{df}{dx}g+f\frac{dg}{dx}$ **ID1**

So we have to remember that nabla must be applied to all the fields in the bracket.
How?

By adding dots to each field and rewriting the expression as a sum:

$$\nabla\cdot\left(\phi\overline{A}\right)=\nabla\cdot\left(\underset{\cdot}{\phi}\overline{A}\right)+\nabla\cdot\left(\phi\underset{\cdot}{\overline{A}}\right)$$

Then we must remember that nabla will be applied only to the field with the dot.
Now we can rewrite the expression using vector algebra rules:

$$\nabla\cdot\left(\phi\overline{A}\right)=\underbrace{\nabla\cdot\left(\underset{\cdot}{\phi}\overline{A}\right)+\nabla\cdot\left(\phi\underset{\cdot}{\overline{A}}\right)}=\underbrace{\overline{A}\cdot\nabla\phi+\phi\nabla\cdot\overline{A}}$$ **ID2**

*rewriting the expression using vector algebra*

$$\overline{n}\cdot\left(\underset{\cdot}{c}\overline{a}\right)+\overline{n}\cdot\left(c\underset{\cdot}{\overline{a}}\right)=$$

$$\overline{n}\cdot\left(\overline{a}\right)\underset{\cdot}{c}+\overline{n}\cdot\left(\underset{\cdot}{\overline{a}}\right)c=$$

$$\overline{a}\cdot\overline{n}\,\underset{\cdot}{c}+\overline{n}\cdot\underset{\cdot}{\overline{a}}\,c=\overbrace{\overline{a}\cdot\overline{n}\,\underset{\cdot}{c}+c\,\overline{n}\cdot\underset{\cdot}{\overline{a}}}$$

*EXERCISE: prove that* $\nabla\times\left(\phi\overline{A}\right)=\left(\nabla\phi\right)\times\overline{A}+\phi\nabla\times\overline{A}$ **ID3**

# NABLARÄKNING

To correctly perform the nabla calculation you need three steps.

*We want to calculate the following expression:* $\nabla\cdot\cdot\left(\phi,\bar{A},\psi,\bar{B},\ldots\right)$

*Where* $\nabla\cdot\cdot$ *can be:* $\nabla$ *(gradient) or* $\nabla\cdot$ *(divergence) or* $\nabla\times$ *(curl)*

STEP 1 Rewrite the expression as a sum with N terms, where N is the number of (scalar or vector) fields in the expression. Every term in the sum must be identical to the original expression, but the *i-th* field in the *i-th* term must have a dot. Then, the nabla operator will be applied only to the field with the "dot".

$$\nabla\cdot\cdot\left(\phi,\bar{A},\psi,\bar{B},\ldots\right)=\nabla\cdot\cdot\left(\underset{\cdot}{\phi},\bar{A},\psi,\bar{B},\ldots\right)+\nabla\cdot\cdot\left(\phi,\underset{\cdot}{\bar{A}},\psi,\bar{B},\ldots\right)+$$

$$\nabla\cdot\cdot\left(\phi,\bar{A},\underset{\cdot}{\psi},\bar{B},\ldots\right)+\nabla\cdot\cdot\left(\phi,\bar{A},\psi,\underset{\cdot}{\bar{B}},\ldots\right)+\ldots$$

STEP 2 The nabla can now be formally considered as a vector. Each term must be rewritten using vector algebra rules in order that only the field with the "dot" will appear after the nabla.

STEP 3 Finally, you can remove the "dot".

(But remember that the nabla is NOT a vector)

# NABLARÄKNING: EXAMPLES

Prove ID4: $\nabla\cdot\left(\overline{A}\times\overline{B}\right)=\overline{B}\cdot\left(\nabla\times\overline{A}\right)-\overline{A}\cdot\left(\nabla\times\overline{B}\right)$ **ID4**

$$\nabla\cdot\left(\overline{A}\times\overline{B}\right)=\nabla\cdot\left(\underset{\cdot}{\overline{A}}\times\overline{B}\right)+\nabla\cdot\left(\overline{A}\times\underset{\cdot}{\overline{B}}\right)=$$

*Now nabla can be treated as vector.*

*since:* $\overline{n}\cdot\left(\overline{A}\times\overline{B}\right)=\overline{B}\cdot\left(\overline{n}\times\overline{A}\right)=-\overline{A}\cdot\left(\overline{n}\times\overline{B}\right)$

*We obtain:*

$$=\overline{B}\cdot\left(\nabla\times\underset{\cdot}{\overline{A}}\right)-\overline{A}\cdot\left(\nabla\times\underset{\cdot}{\overline{B}}\right)=\overline{B}\cdot rot\,\overline{A}-\overline{A}\cdot rot\,\overline{B}$$

Prove ID7: $\nabla\times\left(\nabla\phi\right)=0$ **ID7**

$$\nabla\times\left(\nabla\phi\right)=\nabla\times\left(\nabla\underset{\cdot}{\phi}\right)=$$

*since:* $\overline{n}\times\left(\overline{n}\lambda\right)=\lambda\left(\overline{n}\times\overline{n}\right)=0$

*We obtain:*

$$=\nabla\times\left(\nabla\underset{\cdot}{\phi}\right)=0$$

Prove ID9: $\nabla\times\left(\nabla\times\overline{A}\right)=\nabla\left(\nabla\cdot\overline{A}\right)-\nabla^2\overline{A}$ **ID9**

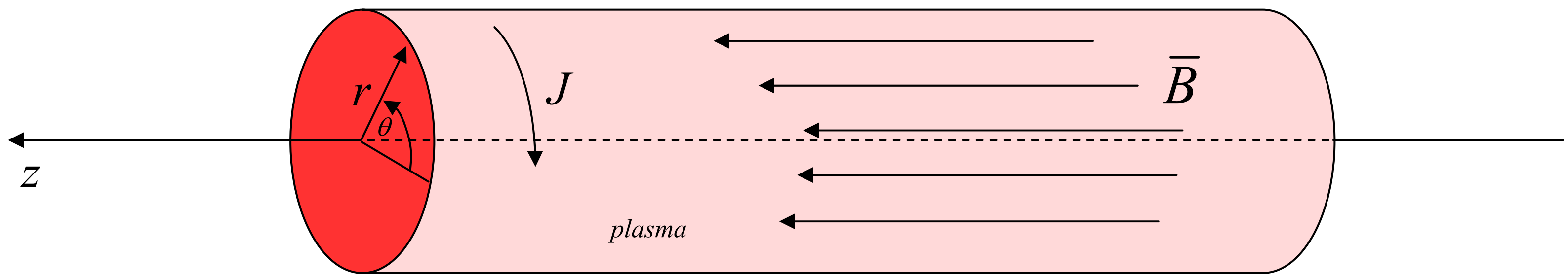
$$\nabla\times\left(\nabla\times\overline{A}\right)=\nabla\times\left(\nabla\times\underset{\cdot}{\overline{A}}\right)=$$

*since:* $\overline{n}\times\left(\overline{n}\times\overline{c}\right)=\overline{n}(\overline{n}\cdot\overline{c})-\overline{c}(\overline{n}\cdot\overline{n})$

*We obtain:*

$$=\nabla\left(\nabla\cdot\underset{\cdot}{\overline{A}}\right)-\left(\nabla\cdot\nabla\right)\underset{\cdot}{\overline{A}}=\nabla\left(\nabla\cdot\overline{A}\right)-\nabla^2\overline{A}$$

# TARGET PROBLEM




$$grad\ p = \frac{1}{\mu_0}\left(rot\bar{B}\right)\times\bar{B}$$

but $\bar{a}\times\left(\bar{n}\times\bar{b}\right)=\bar{n}(\bar{a}\cdot\bar{b})-\bar{b}(\bar{a}\cdot\bar{n})$

$$\nabla p = \frac{1}{\mu_0}\left(\nabla\times\bar{B}\right)\times\bar{B}$$

$$\left(\nabla\times\underset{\cdot}{\bar{B}}\right)\times\bar{B} = -\bar{B}\times\left(\nabla\times\underset{\cdot}{\bar{B}}\right) = -\nabla(\bar{B}\cdot\underset{\cdot}{\bar{B}})+\left(\bar{B}\cdot\nabla\right)\underset{\cdot}{\bar{B}} =$$

$$= -\frac{1}{2}\nabla B^2 + \left(\bar{B}\cdot\nabla\right)\bar{B}$$

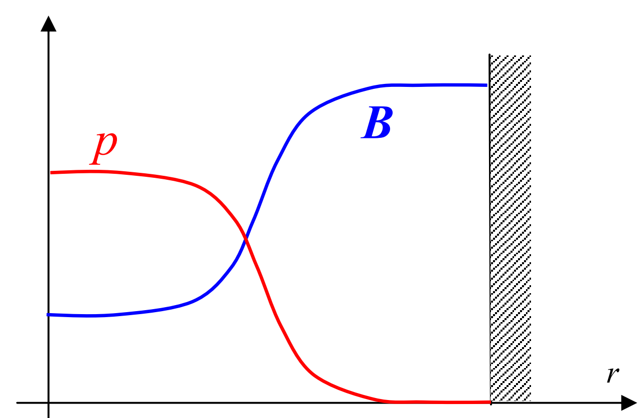
$$\nabla\bar{B}^2 = \nabla\left(\bar{B}\cdot\bar{B}\right) = \nabla\left(\underset{\cdot}{\bar{B}}\cdot\bar{B}\right)+\nabla\left(\bar{B}\cdot\underset{\cdot}{\bar{B}}\right) = 2\nabla(\bar{B}\cdot\underset{\cdot}{\bar{B}})$$

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}\left(\bar{B}\cdot\nabla\right)\bar{B}$$

*particle pressure* *magnetic pressure* *Forces due to bending and parallel compression of the field*

*In our case field lines are straight and parallel*

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)=0 \quad\Rightarrow\quad p+\frac{B^2}{2\mu_0} = constant$$



# A BIT OF HISTORY…

**Why the word "nabla"?**

The theory of nabla operator was developed by Tait (a Maxwell co-worker).
It was one of his most important achievements.
But he was also a good musician in playing an old assyrian instrument similar to an harp.
The name of this instrument in greek is nabla.

The name nabla for the operator was suggested humorously by James Clerk Maxwell

**Why the symbol $\nabla$?**
Here is the nabla instrument…

# WHICH STATEMENT IS WRONG?

1- grad, div and rot can be expressed with the help of $\nabla$ (yellow)

2- $\nabla$ is a vector (red)

3- $\nabla \times (\nabla \phi) = 0$ (green)

4- $\nabla \cdot (\nabla \times \bar{A}) = 0$ (blue)

# INTEGRALSATSER

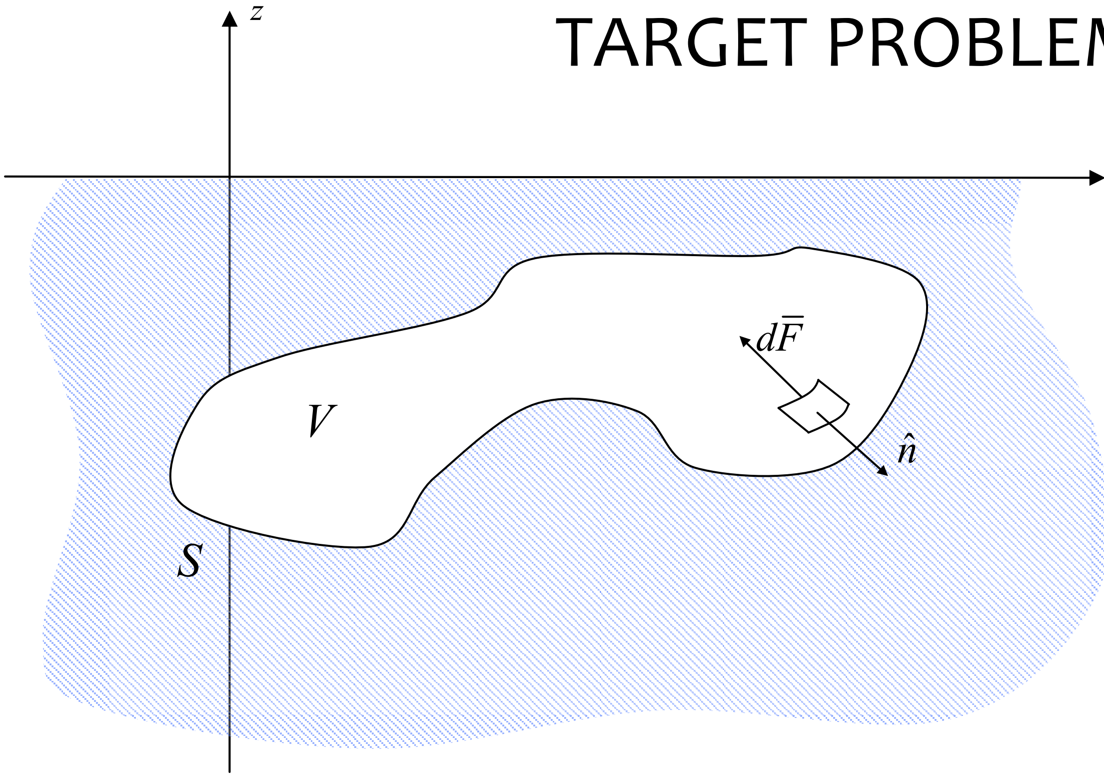
# TARGET PROBLEM

- A body is floating in the water
- What is the force that makes it floating?

- We can use the Arkimedes principle.

- But how does the Arkimedes principle work?

# TARGET PROBLEM




$$d\bar{F} = -p\hat{n}dS$$

*where p [N/m²] is the pressure*

$$\bar{F} = \int d\bar{F} = \oiint_S (-p\hat{n}dS) = -\oiint_S pd\bar{S}$$

How to continue?

Apply Guass theorem? $\oiint_S \bar{A} \cdot d\bar{S} = \iiint_V div\bar{A}dV$

But $\bar{A}$ is vector, while $p$ is a scalar!

In previous lessons we saw that:

$$\int_P^Q \nabla \phi \cdot d\overline{r} = \phi(Q) - \phi(P) \qquad (1)$$

$$\iint_S \nabla \times \overline{A} \cdot d\overline{S} = \oint_L \overline{A} \cdot d\overline{r} \qquad \textit{(Stokes)} \qquad (2)$$

$$\iiint_V \nabla \cdot \overline{A} dV = \oiint_S \overline{A} \cdot d\overline{S} \qquad \textit{(Gauss)} \qquad (3)$$

What do they have in common?
They all express the integral of the derivative of a function in terms of the values of the function at the integration domain boundaries.

In this sense, theorems (1), (2) and (3) are a generalization of:

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

We can further generalize the Gauss theorem :

$$\oiint_S d\overline{S}(\ldots) = \iiint_V dV \nabla(\ldots)$$

*Generalized Gauss theorem*

*where (...) can be substituted with everything that gives a well defined meaning to both sides of the expression.*

$$\oint\!\!\!\oint_S d\overline{S}\,(\ldots) = \iiint_V dV\nabla(\ldots)$$

(A) If we put in $(\ldots) = \cdot\overline{A}$ , we obtain the Gauss theorem *(already proved)*

(B) If we put in $(\ldots) = \phi$ , we obtain: $\oint\!\!\!\oint_S d\overline{S}\phi = \iiint_V dV\nabla\phi$

PROOF

$$\hat{e}_i \cdot \iint_S \phi d\overline{S} = \iint_S \phi\hat{e}_i \cdot d\overline{S} \overset{(Gauss)}{=} \iiint_V \nabla(\phi\hat{e}_i)dV \overset{ID2}{=}$$

$$= \iiint_V \left((\nabla\phi)\cdot\hat{e}_i + \phi\nabla\cdot\hat{e}_i\right)dV = \iiint_V \nabla\phi\cdot\hat{e}_i dV = \hat{e}_i \cdot \iiint_V \nabla\phi dV$$

(C) If we put in $(\ldots) = \times\overline{A}$ , we obtain: $\oint\!\!\!\oint_S d\overline{S}\times\overline{A} = \iiint_V \left(\nabla\times\overline{A}\right)dV$

PROOF

Multiply by $\hat{e}_i$ , use the Gauss theorem and then ID4

We can further generalize also the Stokes' theorem :

$$\oint_L d\bar{r}\,(\ldots) = \iint_S \left(d\bar{S}\times\nabla\right)(\ldots)$$

*Generalized Stokes theorem*

*where (...) can be substituted with everything that gives a well defined meaning to both sides of the expression.*

(A) If $(\ldots) = \cdot\bar{A}$ , we obtain the Stokes theorem *(already proved)*

(B) If $(\ldots) = \phi$ , we obtain: $\oint_L \phi\, d\bar{r} = \iint_S d\bar{S}\times grad\,\phi$

PROOF

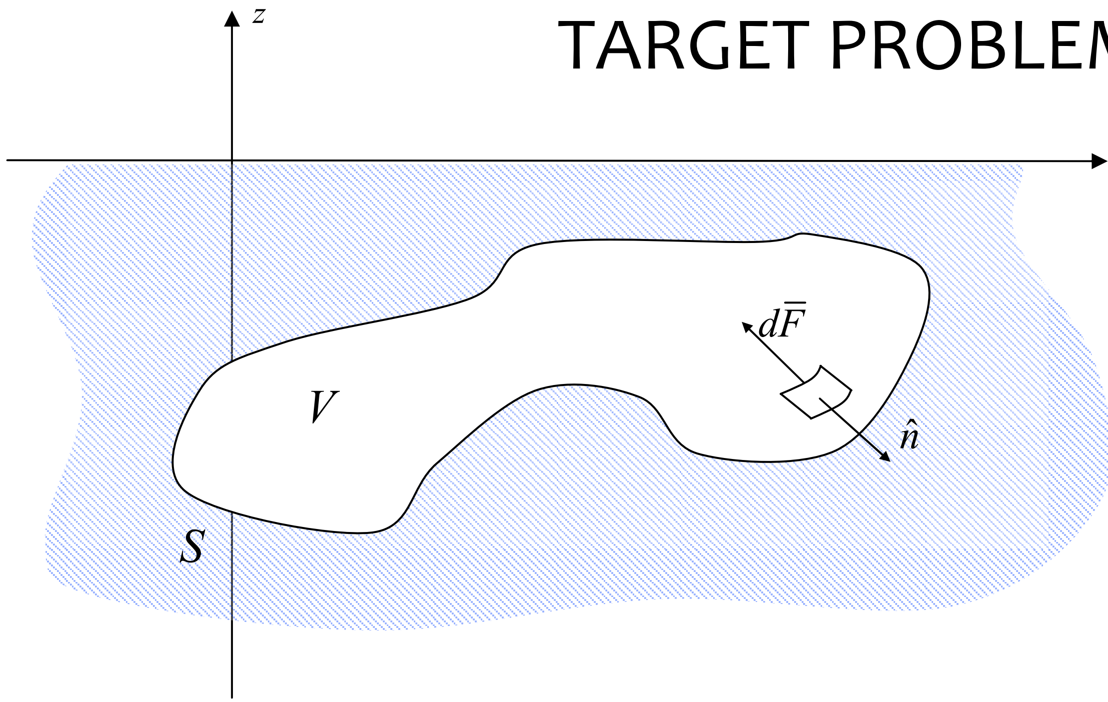
Multiply by $\hat{e}_i$ , use the Stokes theorem and then ID3

(C) If $(\ldots) = \times\bar{A}$, we obtain: $\oint_L d\bar{r}\times\bar{A} = \iint_S \left(d\bar{S}\times\nabla\right)\times\bar{A}$

PROOF

Multiply by $\hat{e}_i$ and use the Stokes theorem.

# TARGET PROBLEM




$$d\bar{F} = -p\hat{n}dS$$

*where p [N/m²] is the pressure*

$$\bar{F} = \int d\bar{F} = \oint\!\!\oint_S (-p\hat{n}dS) = -\oint\!\!\oint_S pd\bar{S}$$

But $\bar{A}$ is vector,
while $p$ is a scalar!

How to continue?
Apply Gauss theorem?

$$\oint\!\!\oint_S \bar{A}\cdot d\bar{S} = \iiint_V div\bar{A}dV$$

We apply the generalized
Gauss theorem, with (…)=$\phi$.

$$\oint\!\!\oint_S \phi d\bar{S} = \iiint_V \nabla\phi dV$$

$$\bar{F} = -\oint\!\!\oint_S pd\bar{S} = -\iiint_V \nabla p\, dV$$

$$p = p_0 - \rho g z$$

$$\nabla p = (0, 0, -\rho g)$$

→

Arkimedes principle

$$\bar{F} = \iiint_V \rho g \hat{e}_z\, dV = \rho g V \hat{e}_z$$

*where $\rho$ is the water density*
*and g the gravitational acceleration*

# WHICH STATEMENT IS WRONG?

1- Gauss and Stokes theorems imply that the integral of the derivative of a function can be expressed as the value of the function at the integration domain boundaries. (yellow)

2- $\int_L \phi d\overline{r}$ is a vector (red)

3- $\iint_S \phi d\overline{S}$ is a vector (green)

4- $\iint_S d\overline{S} \times \overline{A}$ is a scalar (blue)

# INDEXRÄKNING
# (suffix notation)
AND
# CARTESIAN TENSORS

# INDEXRÄKNING

To simplify this expression $\nabla \cdot \left( \overline{A} \times \overline{B} \right)$

we used the "nablaräkning" $= \nabla \cdot \left( \underset{\bullet}{\overline{A}} \times \overline{B} \right) + \nabla \cdot \left( \overline{A} \times \underset{\bullet}{\overline{B}} \right) = \overline{B} \cdot rot\,\overline{A} - \overline{A} \cdot rot\,\overline{B}$

Can we use smarter methods?

YES *(sometimes)* !

These are called "suffix notation methods" ("indexräkning")
and come from the study of tensors.

To understand this method, we start with a *(brief)* look at Cartesian tensors

# PHYSICAL EXAMPLE

## ELECTRICAL CONDUCTIVITY

Ohm's law: $\overline{j} = \sigma \overline{E}$

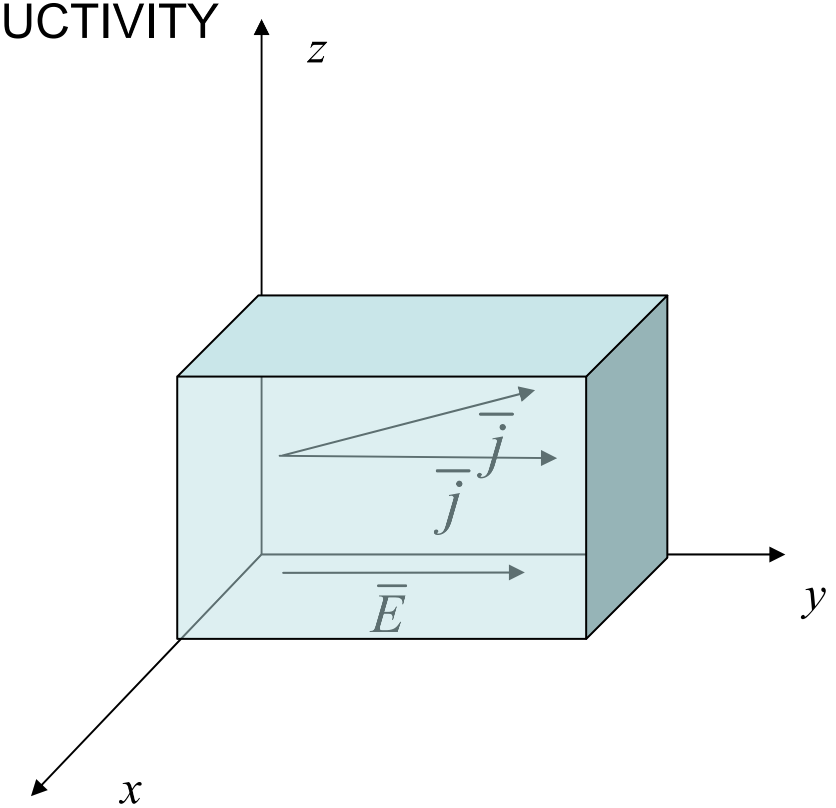
*Current density* — *Electrical conductivity* — *Electric field*

$$If\ \overline{E} = (0, E_y, 0)$$

$$then\ \overline{j} = (0, \sigma E_y, 0)$$

But for many materials this is not true!!

$$\overline{j} = (j_x, j_y, j_z)$$




Is the Ohm's law wrong? NO!

σ is not a scalar

**σ is a cartesian tensor of rank 2**

$$\overline{j} = \sigma \overline{E} \Rightarrow \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$j_i = \sigma_{ik} E_k$$

$$If\ \overline{E} = (0, E_y, 0)$$

$$then\ \overline{j} = (\sigma_{xy} E_y, \sigma_{yy} E_y, \sigma_{zy} E_y)$$

# SUFFIX NOTATION

1- Indices $x, y, z$ can be substituted with $1, 2, 3$

2- Coordinates $x, y, z$ with $x_1, x_2, x_3$.

Examples:

$$A_x = A_1$$

$$(A_x, A_y, A_z) = (A_1, A_2, A_3)$$

$$\hat{e}_x = \hat{e}_1$$

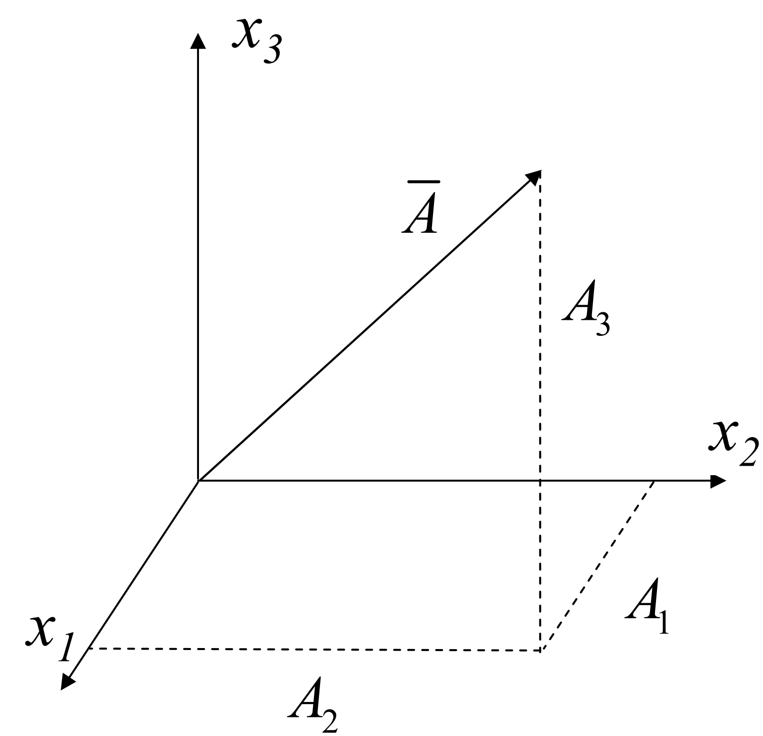
$$\hat{e}_y = \hat{e}_2$$

$$\hat{e}_z = \hat{e}_3$$

$$\frac{\partial \phi}{\partial y} = \partial_2 \phi = \phi_{,2} \qquad \frac{\partial A_x}{\partial y} = A_{1,2}$$

$$\overline{c} = \overline{a} + \overline{b} \quad \Rightarrow \quad \underbrace{c_i = a_i + b_i}$$

*in suffix notation this corresponds to the 3 equations obtained using $i=1,2,3$*




The suffix $i$ is called "free suffix"

The choice of the free suffix is arbitrary: $c_j = a_j + b_j$, $c_m = a_m + b_m$ represent the same equation!

But the same free suffix must be used for each term of the equation

# SUFFIX NOTATION

3- Summation convention:

$$\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1,3} a_i b_i \qquad \Rightarrow \boxed{\bar{a}\cdot\bar{b} = a_i b_i}$$

whenever a suffix is repeated in a single term in an equation, summation from 1 to 3 is implied. The repeated suffix is called *dummy suffix*.

The choice of the dummy suffix is arbitrary: we can write also $\bar{a}\cdot\bar{b} = a_k b_k$

No suffix appears more than twice in any term of the expression:

$$\left(\bar{a}\cdot\bar{b}\right)\left(\bar{c}\cdot\bar{d}\right) = a_i b_i \underbrace{c_j d_j}$$

*we cannot use "i" also here!*

But the ordering of terms is arbitrary: $a_i b_i c_j d_j = c_j a_i d_j b_i = c_k a_m d_k b_m = \left(\bar{a}\cdot\bar{b}\right)\left(\bar{c}\cdot\bar{d}\right)$

Example: $a_k b_h c_k = a_k c_k b_h = \left(\sum_k a_k c_k\right) b_h = (\bar{a}\cdot\bar{c})\bar{b}$

(*free suffix*: $h$; *dummy suffix*: $k$)

*EXERCISE. Write this expression using vectors:* $a_i b_k a_n c_k a_i$

# TENSORS

The Ohm's law is: $\bar{j} = \sigma \bar{E}$

But σ is not a scalar :

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

In suffix notation this can be written very concisely: $j_i = \sigma_{ik} E_k$

σ is a cartesian tensor of rank 2
in the 3-D space.
And it has $3^2$ components

*the rank is the number of suffixes*

A tensor of rank M
in the n-D space has $n^M$ components

$t_{ij}$ is a tensor of rank 2 and can be regarded as a matrix
if it is defined in the 2D space, then *i,j=1,2* and it has $2^2$ components
in the 3D space, then *i,j=1,2,3* and it has $3^2$ components
in the 4D space, then *i,j=1,2,3,4* and it has $4^2$ components
…

$t_m$ is a tensor of rank 1 and can be regarded as a vector

A tensor is "Cartesian" if the coordinate system is Cartesian

# The Kronecker delta

The **Kronecker delta** is a tensor of rank 2 defined as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases}$$

*It can be visualized as a nxn identity matrix (where n is the dimension of the space)*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Some properties of the Kronecker delta:

$$\boxed{\delta_{ii} = 3}$$

$$\delta_{ii} = \sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

*summation convention*

$$\boxed{\delta_{km} a_m = a_k}$$

$$\delta_{km} a_m = \sum_m \delta_{km} a_m = a_1 \delta_{k1} + a_2 \delta_{k2} + \ldots + a_m \delta_{km} + \ldots = a_k$$

$$\boxed{\delta_{km} l_{jm} = l_{jk}}$$

$$l_{jm} \delta_{km} = \sum_m l_{jm} \delta_{km} = \underbrace{l_{j1} \delta_{k1} + l_{j2} \delta_{k2} + \ldots + l_{jm} \delta_{km} + \ldots}_{\text{all zeros, unless } k=m} = l_{jk}$$

*summation convention*

*all zeros, unless k=m*

# The alternating tensor

(Levi-Civita tensor or permutationssymbolen)

The **alternating tensor** $\varepsilon_{ijk}$ *(a tensor of rank 3)* is defined as:

$$\varepsilon_{ijk} = \hat{e}_i \cdot \left( \hat{e}_j \times \hat{e}_k \right) = \begin{cases} 0 & if\ any\ of\ i, j, k\ are\ equal \\ +1 & if\ (i,j,k) = (1,2,3)\ or\ (2,3,1)\ or\ (3,1,2) \quad (even\ \ permutation\ of\ 1,2,3) \\ -1 & if\ (i,j,k) = (1,3,2)\ or\ (2,1,3)\ or\ (3,2,1) \quad (odd\ \ permutation\ of\ 1,2,3) \end{cases}$$

The alternating tensor can be used to express the cross product: $\left( \overline{a} \times \overline{b} \right)_i = \varepsilon_{ijk} a_j b_k$

PROOF:

$$\left( \overline{a} \times \overline{b} \right)_i = \hat{e}_i \cdot \left( \overline{a} \times \overline{b} \right) = \hat{e}_i \cdot \left[ \left( a_j \hat{e}_j \right) \times \left( b_k \hat{e}_k \right) \right] = \hat{e}_i \cdot \left( \hat{e}_j \times \hat{e}_k \right) a_j b_k = \varepsilon_{ijk} a_j b_k$$

EXAMPLE FOR THE $x$ COMPONENT *(i=1)*:

$$\left( \overline{a} \times \overline{b} \right)_1 = a_2 b_3 - a_3 b_2$$

$$\varepsilon_{1jk} a_j b_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{1jk} a_j b_k = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2$$

Some properties:

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$ *(even permutations does NOT change the sign)*

$$\varepsilon_{ijk} = -\varepsilon_{jik}$$ *(odd permutations change the sign)*

$$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$ ← Very useful to simplify expressions involving two cross products

# GRADIENT, DIVERGENCE AND CURL IN SUFFIX NOTATION

**GRADIENT**

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right) = \left(\phi_{,1}, \phi_{,2}, \phi_{,3}\right)$$

So, the component $i$ of the gradient is: $$\boxed{\left(\nabla\phi\right)_i = \phi_{,i}}$$

**DIVERGENCE**

$$\nabla\cdot\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = \sum_i A_{i,i} = A_{i,i}$$

So, the divergence is: $$\boxed{\nabla\cdot\overline{A} = A_{i,i}}$$

**CURL**

$$\left(\nabla\times\overline{A}\right)_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} =$$

$$A_{3,2} - A_{2,3} = \varepsilon_{123}A_{3,2} + \varepsilon_{132}A_{2,3} = \varepsilon_{1jk}A_{k,j}$$

So, the component $i$ of the curl is: $$\boxed{\left(\nabla\times\overline{A}\right)_i = \varepsilon_{ijk}A_{k,j}}$$

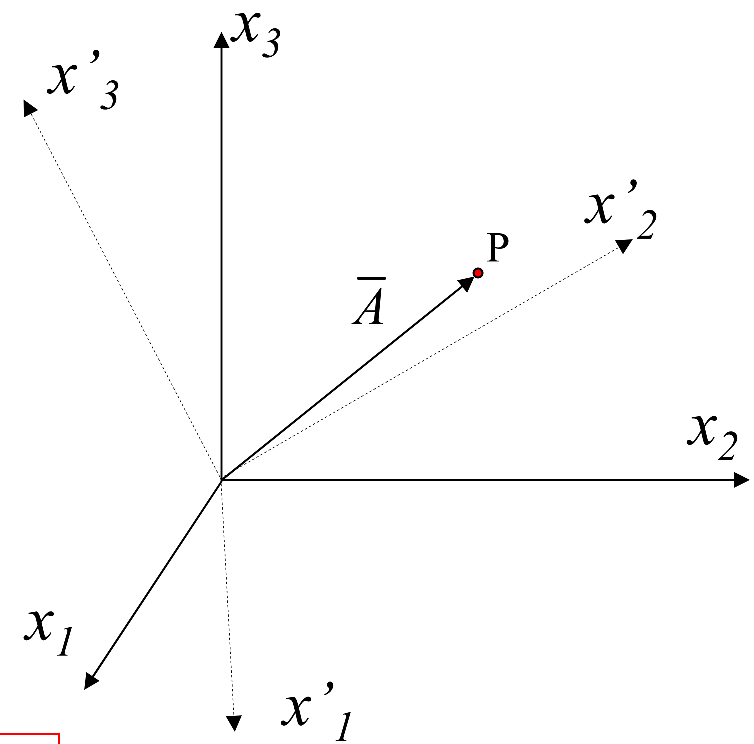
# CARTESIAN TENSORS (the definition)

Assume that the matrix R defines a rotation in a Cartesian coordinate system

In the new coordinate system the vector $\overline{A}$ is:

$$\overline{A}' = R\overline{A}$$

and in suffix notation is: $A'_i = R_{ik} A_k$




A Cartesian tensor T of order M (or rank M) is: a quantity in the 3D Euclidean space that has M indices and $3^M$ components

$$\underbrace{T_{i,j,\ldots,o}}_{M\ indexes} \qquad i, j, \ldots o = 1, 2, 3$$

and which under a rotation of coordinates $R_{ij}$ transforms as:

$$T'_{i,j,\ldots,o} = R_{i,p} R_{j,q} \ldots R_{o,w} T_{p,q,\ldots,w}$$

# “Nablaräkning” and “Indexräkning”

*use of tensors in the calculation of nabla expressions*

Calculate: $\nabla \cdot \left( \overline{a} \times \overline{r} \right)$ where $\overline{r} = \left( x, y, z \right)$ and $\overline{a}$ is constant

## 1- Nablaräkning

$\overline{n} \cdot \left( \overline{a} \times \overline{b} \right) = \overline{a} \cdot \left( \overline{b} \times \overline{n} \right)$

$$\nabla \cdot \left( \overline{a} \times \overline{r} \right) = \nabla \cdot \left( \underset{\bullet}{\overline{a}} \times \overline{r} \right) + \nabla \cdot \left( \overline{a} \times \underset{\bullet}{\overline{r}} \right) = 0 + \overline{a} \cdot \left( \underset{\bullet}{\overline{r}} \times \nabla \right) = -\overline{a} \cdot \underbrace{\left( \nabla \times \overline{r} \right)}_{=0} = 0$$

$\overline{a}$ *is a constant*

## 2- Indexräkning

$$\nabla \cdot \left( \overline{a} \times \overline{r} \right) = \left( \varepsilon_{ikl} a_k r_l \right)_{,i} = \varepsilon_{ikl} \left( a_{k,i} r_l + a_k r_{l,i} \right) = \varepsilon_{ikl} a_k r_{l,i} = 0$$

$r_{l,i} \neq 0 \;\; only \;\; if \;\; l = i$

$If \;\; l = i \;\; then \;\; \varepsilon_{ijk} = 0$