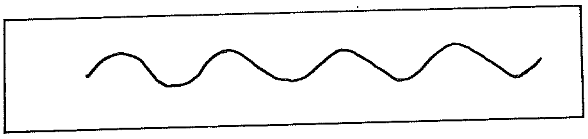
Fourier Analysis

Hermann von Helmholtz
1821–1894

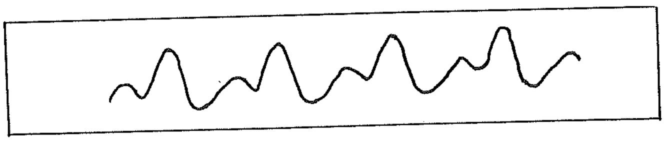
TO GIVE A PERFORMANCE of Verdi's opera *Aida,* one could do without brass and woodwinds, strings and percussion, baritones and sopranos; all that is needed is a complete collection of tuning forks, and an accurate method for controlling their loudness.

This is an application to acoustics of "Fourier's theorem," one of the most useful facts in many branches of physics and engineering. A physical "proof" of the theorem was given by Hermann von Helmholtz when he demonstrated the production of complex musical sounds by suitable combinations of electrically driven tuning forks. (Nowadays, devices of this kind are called electronic music synthesizers.)

In mathematical terms, each tuning fork gives off a vibration whose graph as a function of time is a sine wave:

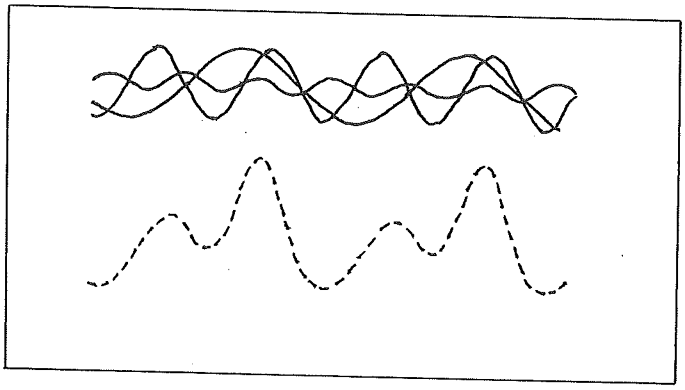


The distance from one peak to the next, the wave-length or the period, would be $\frac{1}{264}$ of a second if the note were middle C. The height of each peak is the amplitude, and roughly measures the loudness. The physical basis of any musical sound is a periodic variation in air pressure whose graph might be a curve like this:



Fourier's theorem says, in graphical terms, that a curve like the one just shown can be obtained by adding up graphs like the first one:

A combination of three pure tones whose frequency ratios are small whole numbers.



In analytic terms, the theorem says that if y is a periodic function that repeats, say, 100 times a second, then y has an expansion like

$$y = 7 \text{ sine } 200\pi t + 0.3 \text{ sine } 400\pi t + 0.4 \text{ sine } 600\pi t + \cdots$$

In each term, the time t is multiplied by 2π times the frequency. The first term, with frequency 100, is called the fundamental, or first harmonic; the higher harmonics all have frequencies that are exact multiples of 100. The coefficients 7, 0.3, 0.4, and so on have to be adjusted to suit the particular sound which we have called "y." The three dots at the end means that the expansion continues indefinitely; the more terms that are included, the more nearly is the sum equal to y.

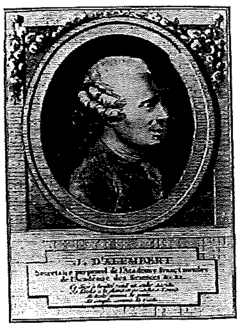
What if y is not periodic—does not repeat itself no matter how long we wait? In that case, we can think of y as the limit of a sequence of functions with longer and longer periods—(which means smaller and smaller frequencies). Fourier's theorem would then require a sum which includes *all* frequencies, not just multiples of a given fundamental frequency. The expansion is then called a Fourier integral instead of a Fourier series.

Once we have translated the theorem from physical to mathematical terms, we have a right to ask for a statement and a proof which meet mathematical standards. What precisely do we allow y to be as a mathematical function? What precisely do we mean by the sum of an infinite series? These questions, raised by the practical demands of Fourier analysis, have taxed the efforts of every great analyst since Euler and Bernoulli; they are still receiving new answers today.

One new answer, a very practical one, is an efficient and ingenious technique for numerically carrying out Fourier analysis on a digital computer. A famous paper by J. W. Cooley and J. W. Tukey in 1965 exploited the binary notation inherent in today's machine computations to make a radical saving in computing time. By taking maximum advantage of the symmetry properties of sine waves, they reduced the number of operations required to find the Fourier expansion of a function given at N data points from N^2 operations to $(2N)$ times the (logarithm of N to the base 2). This reduction was enough to mean that in many applications the effective computational use of Fourier expansions became feasible for the first time. It was reported, for example, that for $N = 8192$, the computations took about five seconds on an IBM 7094; conventional procedures took half an hour.

The origin of Fourier series actually goes back to a problem closely related to the musical interpretation of Fourier analysis with which we began. The problem is that of the motion of a vibrating string.

Waves on Strings

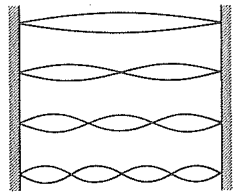


Jean Le Rond d'Alembert
1717–1783

The "wave equation" which governs the vibration of a string was derived in 1747 by d'Alembert. He also found a solution of the equation, in the form of the sum of two traveling waves, of identical but "arbitrary" form, one moving to the right and one moving to the left. Now, if the string is initially at rest (zero velocity), its future motion is determined completely by its initial displacement from equilibrium. Thus there is one arbitrary function in the problem (the one which gives the initial position of the string before it is released) and there is one arbitrary function in d'Alembert's solution (the one which gives the shape of the traveling wave). D'Alembert therefore considered that he had given the general solution of the problem.

However, it is essential to understand that d'Alembert and his contemporaries meant by "function" what nowadays would be called a "formula" or "analytic expression." Euler pointed out that there is no physical reason to require that the initial position of the string is given by a single function. Different parts of the string could very well be described by different formulas (line segments, circular arcs, and so on) as long as they fitted together smoothly. Moreover, the traveling wave solution could be extended to this situation. If the shape of the traveling waves matched the shape of the initial displacement, then Euler claimed the solution was still valid, even though it was not given by a single function but by several, each valid in a different region. The point is that for Euler and d'Alembert every function had a graph, but not every graph represented a single function. Euler argued that any graph (even if not given by a function) should be admitted as a possible initial position of the string. D'Alembert did not accept Euler's physical reasoning.

In 1755 Daniel Bernoulli joined the argument. He found another form of solution for the vibrating string, using "standing waves." A standing wave is a motion of the string in which there are fixed "nodes" which are stationary; between the nodes each segment of the string moves up and down in unison. The "principal mode" is the one without nodes, where the whole string moves together. The "second harmonic" is the name given to the motion with a single node at the mid-point. The "third harmonic" has two equally-spaced nodes, and so on. At any instant, in each of these modes, the string has the form of a sine curve, and at any fixed point on the string the motion in time is given by a cosine function of time. Each "harmonic" thus corresponds to a pure tone of music. Bernoulli's method was to solve the general problem of the vibrating string by summing an infinite number of standing waves. This required that the initial displacement be the sum of an infinite number of sine functions. Physically it meant that any sound produced by the string could be obtained as a sum of pure tones.



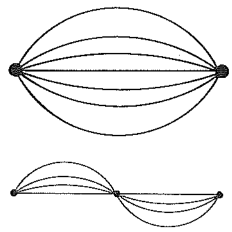
The first four modes of vibration of a string fixed at both ends.

Just as d'Alembert had rejected Euler's reasoning, now Euler rejected Bernoulli's. First of all, as Bernoulli acknowledged, Euler himself had already found the standing wave solution in one special case. Euler's objection was to the claim that the standing-wave solution was *general*—applicable to all motions of the string. He wrote,

> For consider that one has a string which, before release, has a shape which can't be expressed by the equation
>
> $$y = \alpha \sin(\pi x/a) + \beta \sin(2\pi x/a) + \cdots .$$
>
> There are none who doubt that the string, after a sudden release, will have a certain movement. It's quite clear that the figure of the string, the instant after release, will also be different from this equation, and even if, after some time, the string conforms to this equation, one cannot deny that before that time, the movement of the string was different from that contained in the consideration of Bernoulli.

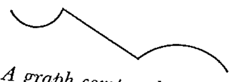
Bernoulli's method involved representing the initial position as an infinite sum of sine functions. Such a sum is a concrete analytic expression which Euler would have regarded as a *single function,* and therefore, to his way of thinking, could not represent an initial position composed of several distinct functions joined together. Moreover, it seemed evident to Euler that the sine series couldn't even represent an arbitrary *single* function, for its ingredients are all periodic and symmetric to the origin. How then could it equal a function which lacked these properties?

Bernoulli did not yield his ground; he maintained that since his expansion contained an infinite number of undetermined coefficients, these could be adjusted to match an arbitrary function at infinitely many points. This argument today seems feeble, for equality at an infinite number of points by no means guarantees equality at *every* point. Nevertheless, as it turned out, Bernoulli was closer to the truth than Euler.



Standing waves on a string; principal mode and second harmonic.

Euler returned to the subject of trigonometric series in



A graph composed of two circular arcs and one straight line. To Euler, this graph is not a graph of one function, but of three. To Fourier and Dirichlet, it is the graph of one function which has a Fourier series expansion.

1777. Now he was considering the case of a function which was known to have a cosine expansion,

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots$$

and he wanted to find a convenient formula for the coefficients a_0, a_1, etc. It seems strange that this problem, which we know today can be solved in one line of calculation, had not already been solved by either Bernoulli or Euler. What is even more astonishing, Euler now found the correct formula, but only by a complicated argument involving repeated use of trigonometric identities and two passages to the limit. Once he had arrived at the simple formula for what we now call the "Fourier coefficients," he did notice the easy trick that would have given the answer immediately.

Say that we want the fifth coefficient a_5. Write down the assumed expansion of f with unknown coefficients:

$$f = a_0 + a_1 \cos x + \cdots + a_5 \cos 5x + \cdots$$

Multiply both sides by cos $5x$ and integrate (i.e., average between the limits $x = 0$ and $x = \pi$).

Now, on the right side we have the integral of an infinite series. By mathematicians of Euler's time, it was taken for granted that one may always evaluate such an expression by integrating each term separately and then adding. But if we integrate each term, we discover that a wonderful thing happens. All the integrals but the fifth equal zero! Since we easily compute $\int_0^\pi a_5 (\cos 5x)^2 dx = \frac{\pi a_5}{2}$, we have $\int_0^\pi f(x) \cos 5x\, dx = \frac{\pi a_5}{2}$, and so $a_5 = \frac{2}{\pi} \int_0^\pi f(x) \cos 5x\, dx$. Similar arguments of course work for all the other coefficients.

This beautifully simple argument is based completely on the fact that $\int_0^\pi \cos mx \cos nx\, dx = 0$ if m is different from n. (A similar formula holds for sines). This property of the cosines is nowadays described in the phrase, "The cosines are orthogonal over the interval from 0 to π." The justification for this geometric language (orthogonal means perpendicular) will appear later on in our story.

In order to appreciate Fourier's work, it is essential to understand that Euler believed to the end that only a very special class of functions, given everywhere by a single analytic expression, could be represented by a sine or cosine series. Only in these special cases did he believe his coefficient formula to be valid.

Fourier's use of sines and cosines in studying heat flow was very similar to Bernoulli's method of studying vibrations. Bernoulli's standing wave is a function of two variables (time t and space x) which has the very special property that it factors into a function of space times a function of time. For such a product to satisfy the vibrating string equation, the two factors must both be sines or cosines. The boundary conditions (ends fixed, initial velocity zero) and the length of the string then determine that they will be of the form $\sin nx$ and $\cos mt$.

When Fourier derived his equation for heat conduction, he found that it too had special solutions which were factorable into a function of space times a function of time. In this case the function of time is exponential rather than trigonometric, but if the solid whose heat flow we are studying is rectangular, we again obtain trigonometric functions of space.

Suppose, for instance, that we have a block of metal whose surface is maintained at a fixed temperature. Then physical considerations show that the interior temperature distribution at time $t = 0$ is sufficient to determine the interior distribution at all later times. But this initial temperature distribution can be arbitrary. *Fourier asserted nevertheless that it is equal to the sum of a series of sines and cosines.* In this he was repeating Bernoulli's point of view. But whereas Bernoulli had in mind only those functions which are formed analytically by a single expression, Fourier explicitly included functions (temperature distributions) given piecewise by several different formulas. In other words, he was asserting that the distinction between "function" and graph, which had been implicitly recognized by all previous analysts, was nonexistent; just as every "function" has a graph, so every graph represents a function—its Fourier series! No wonder Lagrange, the eighteenth century analyst par excellence, found Fourier's claim hard to swallow.

How Fourier Calculated

Joseph Fourier
1768–1830

Of course an essential step in Fourier's work was to find the formula for the coefficients in the expansion. Fourier didn't know Euler had already done this, so he did it over. And Fourier, like Bernoulli and Euler before him, overlooked the beautifully direct method of orthogonality which we have just explained. Instead, he went through an incredible computation, that could serve as a classic example of physical insight leading to the right answer in spite of flagrantly wrong reasoning.

He started out by expanding each sine function in a power series (Taylor series), and then rearranging terms, so that the "arbitrary" function f is now represented by a power series. This already is objectionable, for the functions Fourier had in mind certainly have no such expansion in general. Nevertheless, Fourier proceeded to find the coefficients in this nonexistent power-series expansion. In doing so he used two flagrantly inconsistent assumptions, and arrived at an answer involving division by a divergent infinite product (i.e., an arbitrarily large number). The only sensible interpretation one could give to this formula for the power series expansion was that all the coefficients vanish—i.e., the "arbitrary" function is identically zero. "Fourier had no intention whatsoever of drawing that conclusion, and hence proceeded undismayed with the analysis of his formula." This is the comment of Rudolf Langer, in an article to which we are indebted for our synopsis of Fourier's derivation. From this unpromising formula Fourier was able, by dint of still more formal manipulations, ultimately to arrive at the same simple formula that Euler had obtained correctly, and much more easily, thirty years earlier.

It is a tribute to the insight of Legendre, Laplace, and Lagrange that they awarded Fourier the Grand Prize of the Academy despite the glaring defects in his reasoning.

For Fourier's master stroke came *after* he arrived at Euler's formula. At this point he noticed, as Euler had, that the simple formula could have been obtained in one line, by using the orthogonality of the sines. But then he observed further, as no one before him had done, that the final formula for the coefficients, and the derivation by the orthogonality of the sines, remain meaningful for any graph which bounds a definite area—and this, for Fourier, meant any graph at all. He had already computed the Fourier series for a number of special examples. He found numerically in every case that the sum of the first few terms was very close to the actual graph which generated the series. On this basis, he proclaimed that every temperature distribution—or, if you will, every graph, no matter how many separate pieces it consists of—is representable by a series of sines and cosines. It should be clear that while a collection of special examples may carry conviction, it is in no sense a proof as that word was and is understood by mathematicians. "It was, no doubt," says Langer, "partially because of his very disregard for rigor that he was able to take conceptual steps which were inherently impossible to men of more critical genius."

Fourier was right, even though he neither stated nor proved a correct theorem about Fourier series. The tools he used so recklessly give his name a deserved immortality. To make sense out of what he did took a century of effort by men of "more critical genius," and the end is not yet in sight.

What is a Function?

First of all, what about Euler's seemingly cogent objections of half a century before? How was it possible that a sum of periodic functions (sines and cosines) could equal an arbitrary function which happened *not* to be periodic? Very simply. The arbitrary function is given only on a certain range, say from 0 to π. Physically, it represents the initial displacement of a string of length π, or the initial temperature of a rod of length π. It is only in this range that the physical variables are meaningful, and it is precisely in this range that the Fourier series equals the given function.

Peter Gustav Lejeune Dirichlet 1805–1859

It is irrelevant whether the given function may have a continuation outside this range; if it does, it will *not* in general equal the Fourier series there. In other words, it can perfectly well happen that we have two functions which are identical on a certain range, say from 0 to π, and unrelated elsewhere. This is a possibility that seems never to have been considered by d'Alembert, Euler, and Lagrange. It not only made possible the systematic use of Fourier series in applied mathematics; it also led to the first careful and critical study of the notion of function, which in all its ramifications is as fruitful as any other idea in science.

It was Dirichlet (1805–1859) who took Fourier's examples and unproved conjectures and turned them into respectable mathematics. The first prerequisite was a clear, explicit definition of a function. Dirichlet gave the definition which to this day is the most often used. A function $y(x)$ is given if we have *any* rule which assigns a definite value y to every x in a certain set of points. "It is not necessary that y be subject to the same rule as regards x throughout the interval," wrote Dirichlet; "indeed, one need not even be able to express the relationship through mathematical operations . . . It doesn't matter if one thinks of this [correspondence] so that different parts are given by different laws or designates it [the correspondence] entirely lawlessly. . . . If a function is specified only for part of an interval, the manner of its continuation for the rest of the interval is entirely arbitrary."

Was this what Fourier meant by "an arbitrary function"? Certainly not in the sense in which Dirichlet interpreted the phrase, "any rule." Consider the following famous example which Dirichlet gave in 1828: $\phi(x)$ is defined to be 1 if x is rational, $\phi(x) = 0$ if x is irrational. Since every interval, no matter how small, contains both rational and irrational points, it would be quite impossible to draw a graph of this function. Thus, with Dirichlet's definition of a function, analysis has overtaken geometry and left it far behind. Whereas the restricted eighteenth-century concept of function was not adequate to describe such easily visualized curves as on page 260 the nineteenth-century concept of an arbitrary function includes creatures beyond anyone's hope of drawing or visualizing.

It is readily evident that one can hardly expect this 0–1 function of Dirichlet to be represented by a Fourier series. Indeed, since the area under such a "curve" is undefined, and since the Euler coefficients are obtained by integrating (i.e., computing an area), Fourier could not have found even a single term of the Fourier series for this example. But of course, the practical-minded physicist Fourier did not have in mind such perverse inventions of pure mathematics as this.

On the positive side, Dirichlet proved, correctly and rigorously, that if a function f has a graph which contains only a finite number of turning points, and is smooth except for a finite number of corners and jumps, then the Fourier series of f actually has a sum whose value at each point is the same as the value of f at that point. (Assuming that at points where f has a jump, it is assigned a value equal to the average of the values on the left and on the right.)

This is the result that used to be presented in old-fashioned courses on engineering mathematics, on the grounds that any function which ever arises in physics would satisfy "Dirichlet's criterion." It is plausible that any curve which can be drawn with chalk or pen satisfies Dirichlet's criterion. Yet such curves are far from adequate to represent all situations of physical or engineering interest.

Let us stress the meaning of Dirichlet's result. The formula $y(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$ is true in the following sense. If we choose any given value x_0 between 0 and π, then $y(x_0)$ is a number, and the right hand side is a sum of numbers. It is asserted that if enough terms are taken in the series, the sum of numbers is as close as you wish to the value of y at the given point x_0. This is *pointwise* convergence, the seemingly simplest, and in reality the most complicated, of many possible notions of convergence. From a purely mathematical viewpoint, Dirichlet's result was not an end but a beginning. What a mathematician wants is a nice, clear-cut answer—a necessary and sufficient condition, as they say in the trade. Dirichlet's criterion is a sufficient condition, but by no means a necessary one.

Bernhard Riemann (1816–1866) saw that further progress required a more general concept of integration, powerful enough to handle functions with infinitely many discontinuities. For Euler's formula gives the Fourier coefficients of f as integrals of f times a sine wave. If the function f is generalized beyond the intuitive notion of a smooth curve, then the integral of f must also be generalized beyond the intuitive notion of area under a curve. Riemann accomplished such a generalization. Using his "Riemann integral," he was able to give examples of functions violating Dirichlet's conditions, yet still satisfying Fourier's theorem.

The search for necessary and sufficient conditions under which Fourier's theorem should be valid has been long and arduous. For physical applications, one certainly wants to admit functions having jumps. This means we allow f to be discontinuous. We certainly want it to be integrable, since the coefficients are computed by an integration. Now, if the values of f are changed at a single point, or several points, this is not enough to affect the value of the integral (which is the average of f over all the uncountably many points between 0 and π.) Therefore the Fourier coefficients of f are unchanged. This observation shows that pointwise convergence is not the natural way to study the problem, since there can be points where two functions f and g differ, yet f and g can still have the same Fourier expansion. Indeed, it was the attempt to understand which sets of points were irrelevant for the Fourier series that led Georg Cantor to take the first steps in creating his abstract theory of sets.

A more modest and more reasonable request than convergence at *every* point is to ask that the Fourier series of the function f should equal f except possibly on a set so small it is not noticed by the process of integration. Such sets, defined precisely by H. Lebesgue (1875–1941), are called sets of measure zero, and are used to define a still more powerful notion of the integral than Riemann's. One can think of these sets in the following way: if you pick a point between 0 and 1 at random, your chance of landing in any given interval just equals the length of that interval. If your chance of landing on a given set of points is zero, then that set is said to have measure zero.

Henri Lebesgue
1875–1941

The length of a point is zero by definition. If we add the lengths of several points, this sum is also zero. Therefore, a set of finitely many points has measure zero. There are also sets of measure zero with infinitely many points. It is even possible for a set to have measure zero and yet to be "everywhere dense"—i.e., have a representative in every interval, however small. In fact, the set of all rational numbers is precisely such an everywhere dense set of measure zero. Thus from Lebesgue's viewpoint, Dirichlet's 0-1 function *does* have a Fourier expansion—and every coefficient is zero, since the function is zero "almost everywhere," as Lebesgue put it. This is the kind of mathematics that makes "practical" people shudder. What use is a Fourier expansion if it gives the wrong answer, not just at a few isolated points, but on an everywhere dense set?

But even if we are willing to accept convergence only "almost everywhere" (i.e., except on a set of measure zero), we may not get it. In 1926 Kolmogorov constructed an integrable function whose Fourier series diverged *everywhere.* So integrability alone certainly is not a basis for even an "almost everywhere" theory.

Generalized Functions

A different approach, and one very much in the mainstream of modern analysis, is to take the "orthogonality" property of the sine wave much more seriously. If f is a π-periodic function whose *square* is integrable, it follows from the orthogonality of sines that $\frac{1}{\pi}\int_0^\pi f^2 = b_1^2 + b_2^2 + b_3^2 + \cdots$ where the b's are the coefficients in the sine wave expansion of f. (For the proof, evaluate $\int_0^\pi f^2 = \int_0^\pi f \cdot f$ by expanding each factor f in its sine series, multiplying the first series by the second, and integrating term by term. Because of the orthogonality, most of the integrals are zero, and the rest can be evaluated to give the formula.)

The key idea now is to notice that this sum of squares is analogous to that appearing in the Pythagorean theorem of Euclidean geometry.

According to elementary Euclidean geometry, if P is a point with coordinates (x, y) in the plane or (x, y, z) in space, then the vector OP from the origin to P has a length whose square equals

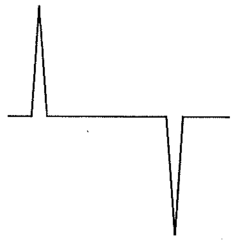
$$\overline{OP}^2 = x^2 + y^2 \quad \text{or}$$
$$\overline{OP}^2 = x^2 + y^2 + z^2 \quad \text{respectively.}$$

This analogy suggests that we think of the function f as a vector in some sort of super-Euclidean space, with rectangular (orthogonal) coordinates b_1, b_2, b_3, etc. Evidently it will be an infinite-dimensional space. Then the "length" of f will have a natural definition as the square root of $\frac{1}{\pi}\int_0^{\pi} f^2$, which is the same as the square root of $b_1^2 + b_2^2 + b_3^2 + \cdots$. The "distance" between two functions f and g would be the "length" of $f - g$.

The space of functions thus defined is called L_2, and is the oldest and standard example of the class of abstract spaces known as Hilbert spaces. The 2 in L_2 comes from the exponent in the squaring operation. The L reminds us that we must integrate with respect to Lebesgue measure. Now we have a new interpretation for convergence of the Fourier series; we ask that the sum of the first 10,000 terms (or 100,000, or 1,000,000 if necessary) should be close to f in the sense of distance in L_2. That is, the difference should give a small number when squared and integrated.

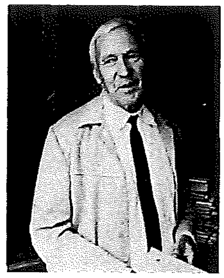
From the Hilbert-space view-point, the subtleties and difficulties of Fourier analysis seem to evaporate like mist. Now the facts are simply proved and simply stated: a function is in L_2 (i.e., is square-integrable) if and only if its Fourier series is convergent in the sense of L_2. (This fact has gone down in history as the Riesz-Fischer theorem.)

There remained, however, the open question as to how bad the pointwise behavior of L_2 functions could be. In view of Kolmogorov's example of an integrable function whose Fourier series diverged everywhere, there was a great sensation when in 1966 Lennart Carleson proved that if a function is *square* integrable, its Fourier series *converges* pointwise almost everywhere. This includes as a special case the new result that a continuous periodic function has a Fourier series that converges almost everywhere. The theory was rounded off when, also in 1966, Katznelson and Kahane showed that for any set of measure zero there exists a continuous function whose Fourier series diverges on that set.



This is a graph of a function which is close to zero in the sense of the Hilbert space L_2, but not in the ordinary sense of distance between curves. The tall spikes are negligible in L_2 because the area they contain is very small.

It is interesting to observe that this modern development really involves a further evolution of the concept of function. For an element in L_2 is not a function, either in Euler's sense of an analytic expression, or in Dirichlet's sense of a rule or mapping associating one set of numbers with another.



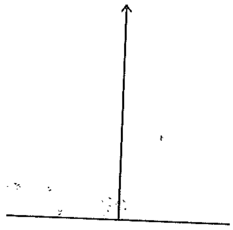
Andrei Kolmogorov
1903–

It is function-like in the sense that it can be subjected to certain operations normally applied to functions (adding, multiplying, integrating). But since it is regarded as unchanged if its values are altered on an arbitrary set of measure zero, it is certainly not just a rule assigning values at each point in its domain.

As we have seen, the development of Fourier analysis in the nineteenth century achieved logical rigor, but at the price of a certain split between the pure and applied viewpoints. This split still exists, but the thrust of much recent and contemporary work is to reunite these two aspects of Fourier analysis.

Norbert Wiener
1894–1964

First of all, the concept of Hilbert space, abstract as it is, provides the foundation of quantum mechanics. It has therefore been an essential topic in applied mathematics for the last fifty years. Moreover, the major expansion of Fourier analysis, in Norbert Wiener's generalized harmonic analysis, and in Laurent Schwartz's theory of generalized functions, is directly motivated by applications of the most concrete kind. For instance, in electrical engineering one often imagines that a circuit is closed instantaneously. Then the current would jump from a value of zero before



Dirac's delta function is zero except at a single point, where it is infinite. In modern analysis, even this very eccentric "function" can be expressed as an infinite series of cosines.

the switch is closed to a value of, say, 1, after the switch is closed. Clearly there is no finite rate of change of current at switch-on time. To put it geometrically, the graph of the current is vertical at $t = 0$. Nevertheless, it is very convenient in calculations to use a fictitious rate of change, which is infinite at $t = 0$ (Dirac's delta function). The theory of generalized functions supplies a logical foundation for using such "impulse" functions or pseudofunctions. This theory permits us to differentiate *any* function as many times as we please; the only trouble is that we must allow the resulting object to be, not a genuine function, but a "generalized function." In a historical perspective, the interesting thing is that the concept of function has had to be widened still further, beyond either Dirichlet or Hilbert.

Furthermore, one of the payoffs for this widening is that in a sense we can return to the spirit of Fourier. For when we construct the Fourier expansion of one of these "generalized functions," we obtain a series or integral which is divergent in any of the senses we have considered. Nevertheless, formal manipulations in the style of Euler or Fourier now often become meaningful and reliable in the context of the new theory.

Thus mathematicians have labored for a century and a half to justify some of Fourier's computations. On the other hand, few physicists and engineers ever felt the need for a justification. (After all, a working gadget or a successful experiment speaks for itself.) Still, they do seem to take some comfort from the license which mathematics has now granted them. In recent applied textbooks, the early pages are now lightly sprinkled with references to Laurent Schwartz as if to justify previously "illicit" computations.

Further Readings. See Bibliography

F. J. Arago; E. T. Bell [1937]; J. W. Dauben; I. Grattan-Guiness; R. E. Langer; G. Weiss

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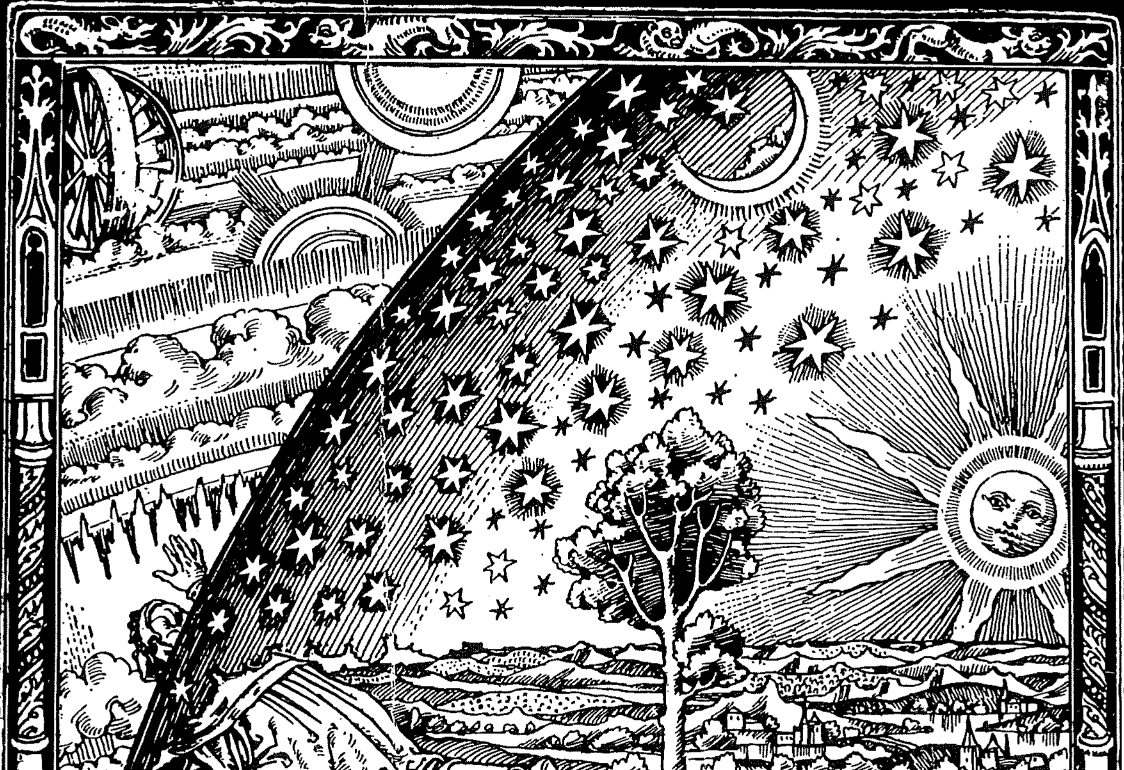

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